Stat 274 - Homework Assignment 1 Solution

1. Suppose $a(t) = 1 + 0.02t + 0.001t^2$

   (a) Find i_3, the effective interest rate for the third year. [0.023946]
   Answer:

   $$i_3 = \frac{a(3) - a(2)}{a(2)} = \frac{1.069 - 1.044}{1.044} = .023946$$

   (b) Find $i_{[3,5]}$, the effective interest rate for the time period from 3 to 5. [0.052385]
   Answer:

   $$i_{[3,5]} = \frac{a(5) - a(3)}{a(3)} = \frac{1.125 - 1.069}{1.069} = .052385$$

   (c) Given an initial deposit of 25, what will be the accumulated amount at time 5? [28.125]
   Answer:

   $$A(5) = 25 * a(5) = 25 * 1.125 = 28.125$$

   (d) Given an initial deposit of 25, what will be the interest earned in the fifth year? [0.725]
   Answer:

   $$I_5 = A(5) - A(4) = 25[a(5) - a(4)] = 25[1.125 - 1.096] = .725$$

2. A loan is made at time 0 at simple interest at an annual rate of 5%.

   (a) In which year is the effective rate 1/23? [4]
   Answer:

   $$\frac{1}{23} = \frac{s}{1 + s(n-1)} = \frac{.05}{1 + .05(n-1)} = \frac{.05}{.95 + .05n}$$
   $$n = 4$$

   (b) What is the effective rate for the interval $[4, 6]$? [0.0833]
   Answer:
   Recall, with simple interest: a(t) = 1 + st

   $$i_{[4,6]} = \frac{a(6) - a(4)}{a(4)} = \frac{1.3 - 1.2}{1.2} = .0833$$

3. Assuming simple interest

(a) With $i = 0.03$, an initial deposit of $K = 2500$, and $A(t) = 3200$, find t. [9.33]
Answer:

$$\begin{aligned} A(t) &= k * a(t) \\ 3200 &= 2500[1 + .03(t)] \\ t &= 9.33 \end{aligned}$$

(b) With an interest rate of 5% and an initial deposit of 3000, find the accumulated value at time 6.5. [3975]
Answer:

$$A(6.5) = 3000 * a(6.5) = 3000[1 + .05(6.5)] = 3975$$

(c) How much would you need to deposit now in order to have 10000 in ten years with simple interest credited at a rate of 8% per year? [5555.55]
Answer:

$$k = \frac{A(t)}{a(t)} = \frac{10000}{1 + .08(10)} = 5555.55$$

(d) Suppose that an initial deposit of 5000 increases to 8000 in 10 years, find the annual interest rate. [0.06]
Answer:

$$\begin{aligned} 8000 &= 5000(1 + 10i) \\ i &= 0.06 \end{aligned}$$

(e) You deposit 100 at time 0. For the first three years you earn 3% each year, for the next two you earn 4%, and for the final year you earn 6%. How much is in the account after the end of the six years? (Note that for simple interest the principal will be the same (100) for each period.) [123]
Answer:

$$100[(1 + 0.03(3)) + (0.04(2)) + (0.06(1))] = 123$$

(f) You deposit 100 at time 0. Accounts earn 2% per year under 125, 3% per year between 125 and 200 and then 4% per year above 200. When will the account have a balance of 225? [43.75]
Answer:

$125 = 100(1 + 0.02t_1)$ $\quad t_1 = 12.5$
$200 - 125 = 75 = 100(.03t_2)$ $\quad t_2 = 25$
$225 - 200 = 25 = 100(.04t_3)$ $\quad t_3 = 6.25$
$t_1 + t_2 + t_3 = 43.75$

4. Rework the previous problem assuming compound interest. [8.35; 4119.57; 4631.93; 0.04812; 125.28; 30.172]
Answer:

(a)

$$2500(1.03)^t = 3200$$
$$t = \frac{log(1.28)}{log(1.03)} = 8.35$$

(b)

$$3000(1.05)^{6.5} = 4119.57$$

(c)

$$k * (1.08)^{10} = 10000$$
$$k = 4631.93$$

(d)

$$5000(1+i)^{10} = 8000$$
$$1 + i = \left(\frac{8000}{5000}\right)^{1/10}$$
$$i = .04812$$

(e)

$$100(1.03)^3(1.04)^2(1.06) = 125.28$$

(f)

$$100(1.02)^{t_1} = 125 \longrightarrow t_1 = 11.268$$
$$125(1.02)^{t_2} = 200 \longrightarrow t_2 = 15.901$$
$$200(1.04)^{t_3} = 225 \longrightarrow t_3 = 3.003$$
$$t_1 + t_2 + t_3 = 30.172$$

5. Suppose that $a(t) = \alpha + \beta t + \gamma t^2$, find the values of α, β, and γ given that $i_1 = 0.05$ and $i_{[0,2]} = 0.12$. [$\alpha = 1$; $\beta = 0.04$; $\gamma = 0.01$]
Answer:

$$a(t) = \alpha + \beta t + \gamma t^2$$
$$a(0) = 1 \longrightarrow \alpha = 1$$
$$\frac{a(1) - a(0)}{a(0)} = \frac{1 + \beta + \gamma - 1}{1} = \beta + \gamma = .05$$
$$\frac{a(2) - a(0)}{a(0)} = \frac{1 + 2\beta + 4\gamma - 1}{1} = 2\beta + 4\gamma = .12$$

Solving, we get $\beta = 0.04$ and $\gamma = .01$

6. Under annually compounding interest with a positive interest rate, the effective interest rate for [8,14] is 2.1 times the effective interest rate for [3,6]. Find the annual interest rate i. [3.23%]
Answer:

$$\begin{aligned} 2.1((1+i)^3 - 1) &= (1+i)^6 - 1 \\ 2.1(1+i)^3 - 2.1 &= (1+i)^6 - 1 \\ 0 &= (1+i)^6 - 2.1(1+i)^3 + 1.1 \end{aligned}$$

Substitute $x = (1+i)^3$

$$\begin{aligned} 0 &= x^2 - 2.1x + 1.1 \\ x &= \frac{2.1 \pm \sqrt{2.1^2 - 4 \cdot 1 \cdot 1.1}}{2} \\ x &= \{1, 1.1\} \\ i &= \{0, 0.03228\} \end{aligned}$$

7. Account A has 100 dollars at time 0 and grows at a simple interest rate of 0.05. Account B has x dollars at time 0 and grows at a compound interest rate of 0.03. At time $t = 9$, accounts A and B are equal. Solve for x. [111.13]
Answer:

$$\begin{aligned} 100(1 + 0.05(9)) &= x(1 + 0.03)^9 \\ x &= 111.13 \end{aligned}$$

8. Richard has 1500 dollars. He wants to have 2500 dollars in 10 years. His bank will pay him simple interest i for 5 years, after which it will pay him compound interest i for 5 more years. Solve for i. You may use software to solve the final equation. [0.0550203]
Answer:

$$\begin{aligned} 2500 &= 1500(1 + 5i)(1 + i)^5 \\ i &= 0.0550203 \end{aligned}$$