Stat 274 — Homework Assignment 4 Solution

1. Which of the following expressions does NOT represent a definition for $a_{\overline{n}|}$?

   (a) $v^n \left[\dfrac{(1+i)^n - 1}{i} \right]$

   (b) $\dfrac{1 - v^n}{i}$

   (c) $v + v^2 + \cdots + v^n$

   (d) $v \left[\dfrac{1 - v^n}{1 - v} \right]$

   (e) $\dfrac{s_{\overline{n}|}}{(1+i)^{n-1}}$

   Answer:

   - For (a), $\dfrac{(1+i)^n - 1}{i} = s_{\overline{n}|}$; $s_{\overline{n}|} v^n = a_{\overline{n}|}$
   - For (b), $\dfrac{1 - v^n}{i}$ is the closed form of $a_{\overline{n}|}$
   - For (c), $v + v^2 + \cdots + v^n$ is the open form of $a_{\overline{n}|}$
   - For (d), $v = \dfrac{1}{1+i}$; $v + vi = 1$; $\dfrac{1-v}{v} = i$; $\dfrac{1}{i} = \dfrac{v}{1-v}$

     Plug that into our original equation and we get $\dfrac{1 - v^n}{i} = a_{\overline{n}|}$
   - For (e), $s_{\overline{n}|} v^n = \dfrac{s_{\overline{n}|}}{(1+i)^n} = a_{\overline{n}|}$

     But we are given $\dfrac{s_{\overline{n}|}}{(1+i)^{n-1}}$, so (e) is the correct answer.

2. You purchase a home with a loan of 100000. You pay 500 per month for the first 10 years. At the end of the ten years, you pay off the remaining balance with 12 monthly payments of X. The annual nominal interest rate (compounded monthly) is 5%, calculate X. [7452.97]
   Answer:

$$
\begin{aligned}
100000 &= 500 a_{\overline{120}|.05/12} + X a_{\overline{12}|.05/12} * v^{120} \\
100000 &= 500 \Big(\frac{1 - v^{120}}{.05/12} \Big) + X \Big(\frac{1 - v^{12}}{.05/12} \Big) v^{120} \\
X &= 7452.97
\end{aligned}
$$

3. Ernie makes deposits of 100 at time 0, and X at time 3. The fund grows at a force of interest $\delta_t = t^2/100$. The amount of interest earned from time 3 to time 6 is also X. Calculate X.

[784.59]
Answer:

$$\left(100e^{\int_0^3 \frac{t^2}{100}dt} + X\right)\left(e^{\int_3^6 \frac{t^2}{100}dt} - 1\right) = X$$
$$\left(100e^{\frac{t^3}{300}\Big|_0^3} + X\right)\left(e^{\frac{t^3}{300}\Big|_3^6} - 1\right) = X$$
$$(109.417 + X)(1.878 - 1) = X$$
$$784.59 = X$$

4. You put 100 in a savings account at the beginning of each year. The annual nominal interest rate is 5%. How many years do you need to contribute to have at least 750? [7]
Answer:

$$100\ddot{s}_{\overline{n}|.05} = 750$$
$$100\left(\frac{(1.05)^n - 1}{.05}\right)\left(1.05\right) = 750$$
$$(1.05)^n = 1.357$$
$$n = 6.259$$

Since payments are only once per year, you'll have to make 7 payments.

5. You contribute X at the beginning of every other year (starting at time 0). The annual nominal interest rate is 6%. In 17 years (at time 17), you have 1000 in your account. Calculate X. [62.88]
Answer:
Over the 17 years, you will make 9 payments. We will use the equivalent two-year interest rate, 12.36%, in our calculations.

$$X\ddot{s}_{\overline{9}|.1236} = 1000(1.06)$$
$$X\left(\frac{(1.1236)^9 - 1}{.1236}\right)\left(1.1236\right) = 1060$$
$$X = 62.8816$$

6. You deposit 4000 at time 0, it grows to 4040 at time 1, then 3000 is withdrawn, the 1040 grows to 1050 at time 2, then 10000 is deposited and that 11050 grows to 12150 at time 3. Calculate the time-weighted annual yield rate. [3.8876%]
Answer:

$$i_t = \left(\frac{4040}{4000} * \frac{1050}{1040} * \frac{12150}{11050}\right)^{1/3} - 1 = .038876$$
$$i_d = \frac{1150}{4000(1) - 3000(2/3) + 10000(1/3)} * \frac{1}{3} = .071875$$

7. A loan can be paid off by two different payment streams. The first is 100 at times 5, 10, and 20 and another 200 at time 15. The second is a single payment of 1000 at time t. Interest is 4.5% annually. Calculate t. [28.166]
Answer:

$$\begin{aligned} 100(v^5 + v^{10} + v^{20}) + 200v^{15} &= 1000v^t \\ .289446 &= v^t \\ 28.166 &= t \end{aligned}$$

8. You pay 10000 for a special annuity. It pays 100 at times 1, 2, 3, 4, ... , 29, 30 (30 total payments). It also pays a lump sum of 10000 at time t. Assuming the annual effective interest rate is 3%, calculate t. [7.3806]
Answer:

$$\begin{aligned} 10000 &= 100a_{\overline{30}|.03} + 10000v^t \\ 10000 &= 100\Big(\frac{1 - v^{30}}{.03}\Big) + 10000v^t \\ 0.803996 &= v^t \\ 7.3806 &= t \end{aligned}$$

9. You pay 10000 for a special annuity. It pays X at times 1,2,3,4,. . ., 29, 30 (30 total payments). It also pays a lump sum of 10000 at time 35. Assuming the annual effective interest rate is 3%, calculate X. [328.88]
Answer:

$$\begin{aligned} 10000 &= Xa_{\overline{30}|.03} + 10000v^{35} \\ 10000 &= X\Big(\frac{1 - v^{30}}{.03}\Big) + 3553.83 \\ 328.888 &= X \end{aligned}$$

10. Starting on your 25th birthday, and continuing through your 60th birthday, you deposit 750 each year on your birthday into a retirement fund earning an annual effective rate of 5%. Immediately after the last deposit, the accumulated value of the fund is transferred into a fund earning an annual effective rate of j. On your 65th birthday, you purchase a 25-year annuity-due paying 580 each month with the balance of the account. The purchase price of the annuity was determined using an annual effective rate of 4%. Calculate j. [9.094%]
Answer:

$$\begin{aligned} 750(s_{\overline{36}|.05})(1 + j)^5 &= 580\ddot{a}_{\overline{300}|1.04^{1/12}-1} \\ 71877.24(1 + j)^5 &= 111,071.37 \\ (1 + j)^5 &= 1.54529 \\ j &= .09094 \end{aligned}$$

11. You save for retirement by depositing 120 at the beginning of each year for 30 years. One year after the last deposit (at time 30), you want to start taking 20 annual (beginning of the year) level withdrawals of X, which will exhaust your savings. Calculate X if the effective interest rate is 5% during the first 30 years and 4% after that. [592.28]
Answer:

$$\begin{aligned}
120\ddot{s}_{\overline{30}|.05} &= X\ddot{a}_{\overline{20}|.04} \\
120\Big(\frac{(1.05)^{30}-1}{.05}\Big)\Big(1.05\Big) &= X\Big(\frac{1-v^{20}}{.04}\Big)\Big(1.04\Big) \\
8371.29 &= 14.13394X \\
X &= 592.28
\end{aligned}$$

12. An estate provides a perpetuity with payments of X at the end of each year. Seth, Susan, and Lori share the perpetuity such that Seth receives the payments of X for the first n years and Susan receives the payments of X for the next m years, after which Lori receives all the remaining payments of X. Write the representation of the difference between the present value of Seth's and Susan's payments using a constant rate of interest. [$X\big[a_{\overline{n}|} - v^n a_{\overline{m}|}\big]$]

Answer: Seth will receive n payments of X starting at time 0. Susan will receive m payments of X starting at time n. The present value of these annuities are:

$$\begin{aligned}
Seth :& Xa_{\overline{n}|} \\
Susan :& Xa_{\overline{m}|}v^n
\end{aligned}$$

The difference is then:

$$\begin{aligned}
&Xa_{\overline{n}|} - Xa_{\overline{m}|}v^n \\
&X\big[a_{\overline{n}|} - v^n a_{\overline{m}|}\big]
\end{aligned}$$

13. Find and work 5 more practice problems. You can find those:

- In the online practice problems
- In the study manuals in the library
- In the book
- Ask the TA's to write one
- In your purchased software (Infinite Actuary, Coaching Actuaries, Actex, etc.)