Stat 274
Theory of Interest

Chapter 3: Annuities

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Types of Annuities

- Annuity-immediate: Stream of payments at the end of each period.
- Annuity-due: Stream of payments at the beginning of each period.
- Perpetuity: Stream of payments without an end time.

An n-year annuity-immediate pays 1 at times $1, 2, 3, \ldots, n$. Those payments are worth (at time 0)

$$v(1) + v(2) + v(3) + \cdots + v(n)$$

When we have compound interest then $v(t) = (1 + i)^{-t} = v^t$ and the present value becomes

$$a_{\overline{n}|i} = v + v^2 + v^3 + \ldots + v^n = \frac{1 - v^n}{i}$$

Future Value

The future value (at time t) of the annuity is denoted $s_{\overline{n}|}$. We have the following general relationships.

$$s_{\overline{n}|} = a(n) a_{\overline{n}|} \qquad \text{and} \qquad a_{\overline{n}|} = v(n) s_{\overline{n}|}$$

When $a(n) = (1+i)^n$,

$$s_{\overline{n}|i} = (1+i)^n a_{\overline{n}|i} \qquad \text{and} \qquad a_{\overline{n}|i} = v^n s_{\overline{n}|i}$$

$$s_{\overline{n}|i} = (1+i)^{n-1} + (1+i)^{n-2} + \cdots + 1 = \frac{(1+i)^n - 1}{i}$$

The value of n end-of-period payments each of amount Q is equal to L,

$$Qa_{\overline{n}|i} = L.$$

Then the amount of each payment is equal to:

$$Q = \frac{L}{a_{\overline{n}|i}}$$

Example

You can make monthly payments of 500 and have an 1800 down payment. You qualify for a 36-month auto loan at a nominal rate of 4.8% convertible monthly. For how much car do you qualify? [18532.94]

Annuities-due

An n-year annuity-due pays 1 at times $0, 1, 2, \ldots, n-1$. Those payments are worth (at time 0)

$$v(0) + v(1) + v(2) + \cdots + v(n-1)$$

When we have compound interest then $v(t) = (1+i)^{-t} = v^t$ and the present value becomes

$$\ddot{a}_{\overline{n}|i} = 1 + v + v^2 + \ldots + v^{n-1} = \frac{1(1-v^n)}{1-v} = \frac{1-v^n}{d}$$

Future Value

The future value (at time t) of the annuity is denoted $s_{\overline{n|}}$. We have the following general relationships.

$$\ddot{s}_{\overline{n|}} = a(n)\ddot{a}_{\overline{n|}} \qquad \text{and} \qquad \ddot{a}_{\overline{n|}} = v(n)\ddot{s}_{\overline{n|}}$$

When $a(n) = (1+i)^n$,

$$\ddot{s}_{\overline{n|}i} = (1+i)^n \ddot{a}_{\overline{n|}i} \qquad \text{and} \qquad \ddot{a}_{\overline{n|}i} = v^n \ddot{s}_{\overline{n|}i}$$

$$\ddot{s}_{\overline{n|}i} = (1+i)^n + (1+i)^{n-1} + \cdots + (1+i) = \frac{(1+i)^n - 1}{d}$$

Examples

1. You deposit 100 into an account at the beginning of each month for 12 months. Calculate the value of that account at the end of the year with a nominal discount rate of 4.8% payable monthly. [1231.79]
2. You owe your brother 1000, but he will accept 30 monthly payments at the beginning of each month at 2.5% nominal interest payable monthly. How big do your payments have to be? [34.35]
3. You just turned 30. You deposit 400 on every birthday (including today) through your 64th. On your 65th birthday you withdraw X and continue to withdraw X on each birthday through your 80th. Assuming you earn 6% per year before 65 and 5% per year after, calculate X. [4152]
4. Repeat the above problem assuming you earn 6% per year before 64 and 5% per year after. [4112.83]

Comparison of due and immediate

You can think of an annuity-immediate as an annuity-due pushed one period in the future.

$$\ddot{a}_{\overline{n}|i} = (1+i)a_{\overline{n}|i} \qquad \text{and} \qquad \ddot{s}_{\overline{n}|i} = (1+i)s_{\overline{n}|i}$$

Also,

$$\ddot{a}_{\overline{n}|i} = a_{\overline{n-1}|i} + 1 \qquad \text{and} \qquad \ddot{s}_{\overline{n}|i} + 1 = s_{\overline{n+1}|i}$$

Examples

Assume an annual rate of 6%.

1. You just turned 30. You want to buy an annuity which will pay 1000 per year on each of your 65th through 80th birthdays. How much is that annuity worth today? [1393.72]
2. You buy an annuity which pays 100 at times 0, 1, 2, . . ., 29. You need to pay for it at time 15. How much will it cost? [3496.75]
3. You buy an annuity which pays 100 at times 0, 1, 2, . . ., 29. You need to pay for it when you die at time 45. How much will it cost? [20,083.56]

Continuous Annuities

What if you receive payments continuously for a whole year?

$$\begin{aligned}
\bar{a}_{\overline{n}|} &= \int_0^n (1+i)^{-t} dt \\
&= \left[\frac{-(1+i)^{-t}}{log(1+i)} \right]_0^n \\
&= \frac{1-(1+i)^{-n}}{log(1+i)} \\
&= \frac{1-(1+i)^{-n}}{\delta}
\end{aligned}$$

Similarly,

$$\bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta}$$

Perpetuities

As we take the limit of $a_{\overline{n}|i}$ as $n \to \infty$ we find,

$$a_{\overline{\infty}|i} = \frac{1}{i}$$

and

$$\ddot{a}_{\overline{\infty}|i} = \frac{1}{d}$$

which is how much you would have to invest to earn 1 every period forever.

Examples

You have three children, two responsible ones (Arnold and Bridget) and one less so (Cameron). You give them an equal inheritance (in value at the time of your death). Arnold and Bridget split an annual payment at the beginning of the first ten years, starting at the moment of death. After 10 years, Cameron receives the both payments in perpetuity. Calculate the annual effective interest rate. [0.116123]

Geometrically Increasing Annuities

The value (at time 0) of an annuity which pays P at time 1 and $P(1+g)^{k-1}$ at time k is:

$$P\left[\frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g}\right] = \frac{P\ddot{a}_{\overline{n}|j}}{1+i} \quad \text{where } j = \frac{i-g}{1+g}$$

If $i = g$ then the time 0 value is:

$$\frac{nP}{1+i}$$

Examples

1. You purchase an annuity-immediate with 25 annual payments. The first payment is 800 and the payments increase by 3% each year. Using an annual interest rate of 7%, calculate the present value of this annuity. [12,284.46]
2. You purchase an annuity-immediate with 25 annual payments. The first payment is 800 and the payments increase by 3% each year. Using an annual interest rate of 3%, calculate the present value of this annuity. [19,417.48]
3. You purchase an annuity-immediate with 25 annual payments. The first payment is 800 and the payments increase by 3% each year. Using an annual interest rate of 1%, calculate the present value of this annuity. [25,306.48]

The value (at time 0) of a perpetuity which pays P at time 1 and $P(1+g)^{k-1}$ at time k is:

$$lim_{n\to\infty}\frac{P\left[1-\left(\frac{1+g}{1+i}\right)^n\right]}{i-g}=\frac{P}{i-g}\quad \text{for } i>g$$

If $g \geq i$, the present value is infinite because the future payments will be worth at least as much as the earlier payments.

Arithmetically Increasing Annuities

Consider an annuity lasting n periods with a payment of $P + Q(j-1)$ at the end of the j^{th} interest period.

$$(I_{P,Q}s)_{\overline{n}|i} = Ps_{\overline{n}|i} + \frac{Q}{i}(s_{\overline{n}|i} - n)$$

Multiplying by v^n,

$$(I_{P,Q}a)_{\overline{n}|i} = Pa_{\overline{n}|i} + \frac{Q}{i}(a_{\overline{n}|i} - nv^n)$$

Arithmetically Increasing Annuities

When $P = Q = 1$, we can simply drop the P and Q,

$$(Is)_{\overline{n}|i} = \frac{s_{\overline{n+1}|i} - (n+1)}{i} = \frac{\ddot{s}_{\overline{n}|i} - n}{i}$$

Again multiplying by v^n,

$$(Ia)_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - nv^n}{i}$$

When $P = n$ and $Q = -1$,

$$(Da)_{\overline{n}|i} = \frac{n - a_{\overline{n}|i}}{i} \qquad \text{and} \qquad (Ds)_{\overline{n}|i} = \frac{n(1+i)^n - s_{\overline{n}|i}}{i}$$

Arithmetically Increasing Annuities-due

Related formulas are available for annuities-due, most of which simply replace i with d.

$$\begin{aligned}
(I_{P,Q}\ddot{s})_{\overline{n}|i} &= P\ddot{s}_{\overline{n}|i} + \frac{Q}{d}(s_{\overline{n}|i} - n) \\
(I_{P,Q}\ddot{a})_{\overline{n}|i} &= P\ddot{a}_{\overline{n}|i} + \frac{Q}{d}(a_{\overline{n}|i} - nv^n) \\
(I\ddot{s})_{\overline{n}|i} &= \frac{s_{\overline{n+1}|i} - (n+1)}{d} = \frac{\ddot{s}_{\overline{n}|i} - n}{d} \\
(D\ddot{a})_{\overline{n}|i} &= \frac{n - a_{\overline{n}|i}}{d}
\end{aligned}$$

1. You purchase an 17-year annuity-immediate which pays 2000 at time 1, 2500 at time two, 3000 at time 3, increasing by 500 each time. What is the present value of this annuity ($i = 0.042$)? [66,008.43]
2. You deposit 600 into an investment account earning an annual effective rate of 6% at the end of each year for 20 years. The interest earned is reinvested at the end of each year in a different account earning 4%. What is the accumulated value at the time of the last investment? [20,800.27]

Taking the limit of $(I_{P,Q}a)_{\overline{n}|i}$ as $n \to \infty$ we find,

$$(I_{P,Q}a)_{\overline{\infty}|i} = \frac{P}{i} + \frac{Q}{i^2}$$

and

$$(I_{P,Q}\ddot{a})_{\overline{\infty}|i} = \frac{P}{d} + \frac{Q}{id}$$

Annuity Values for Non-integral Terms

While so far we have dealt with integral terms, we know that

$$a_{\overline{n}|i} = \frac{1 - v^n}{i}$$

We can actually define this for any positive real number r (along with many other formulas)

$$a_{\overline{r}|i} = \frac{1 - v^r}{i} \qquad \text{and} \qquad s_{\overline{r}|i} = \frac{(1+i)^r - 1}{i}$$

$$\ddot{a}_{\overline{r}|i} = \frac{1 - v^r}{d} \qquad \text{and} \qquad \ddot{s}_{\overline{r}|i} = \frac{(1+i)^r - 1}{d}$$