

Stat 274  
Theory of Interest

Lecture 6: Term Structure and Interest Rate  
Sensitivity

Brian Hartman  
Brigham Young University

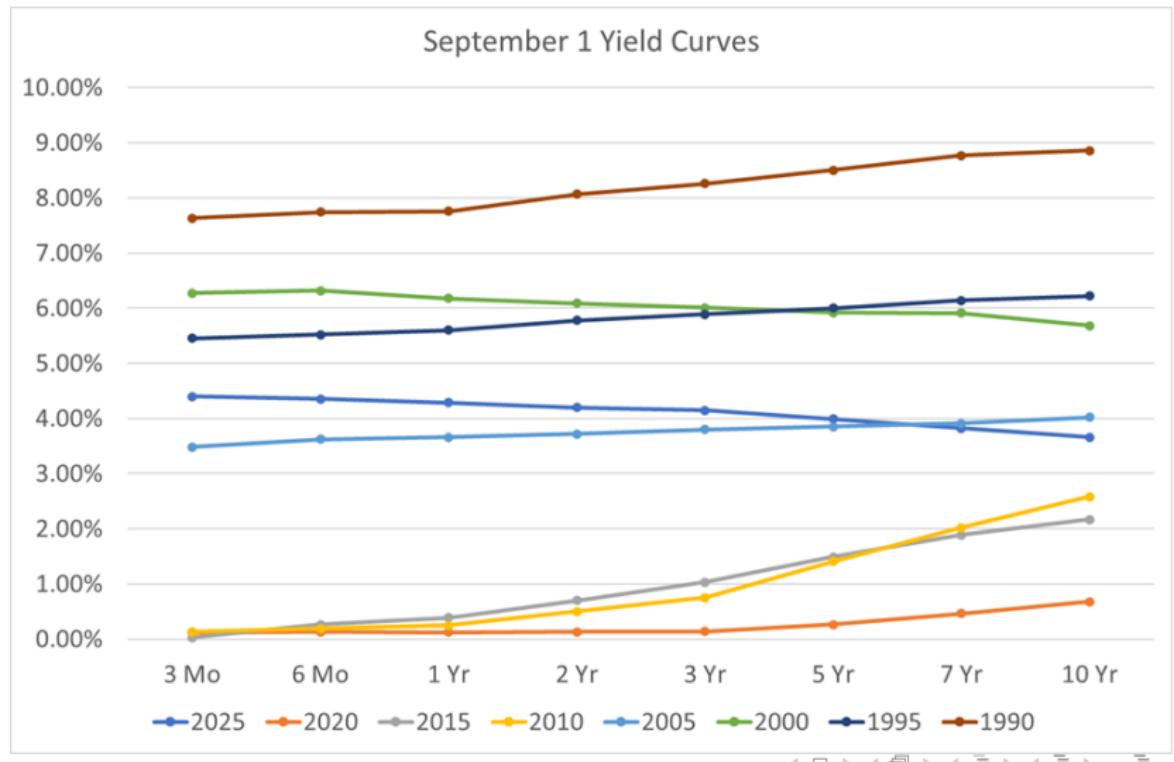
# Yield Curves

- $\nu(t)$  is the current market price for a  $t$ -year zero-coupon bond.
- The  $t$ -year spot rate of interest,  $y_t$ , is the yield per year on a  $t$ -year zero-coupon bond.

$$\nu(t)(1 + y_t)^t = 1 \therefore \nu(t) = \frac{1}{(1 + y_t)^t}$$

- The term structure of the interest rates is the relationship between  $t$  and  $y_t$ .
- It is often presented as a plot of  $y_t$  against  $t$ , known as the yield curve.

# Yield Curve Examples



- Let  $f(t, t + k)$  be the forward rate of interest, the effective annual rate of interest paid between time  $t$  and time  $t + k$ .
- Forward rates use the following no-arbitrage argument.

$$[1 + f(t, t + k)]^k = \frac{(1 + y_{t+k})^{t+k}}{(1 + y_t)^t} = \frac{\nu(t)}{\nu(t + k)}$$

## Examples

$y_1$	$y_2$	$y_3$	$y_4$	$y_5$
0.032	0.035	0.039	0.042	0.043

Using the information above, calculate  $\nu(2)$ ,  $\nu(5)$ , and  $f(1, 2)$ .  
[0.9335, 0.8102, 0.038]

A two-year bond pays annual coupons of 30 and matures for 1,000. Its price is 984. A one-year 1,000 zero-coupon bond sells for 975. Compute the two year spot rate. [3.866%]

# Sensitivity Approximations

Given an interest rate  $i$ , the present value of a set of cash flows is

$$P(i) = \sum_{t \geq 0} C_t (1 + i)^{-t}$$

$$P'(i) = - \sum_{t \geq 0} C_t t (1 + i)^{-t-1}$$

$$P''(i) = \sum_{t \geq 0} C_t t(t+1) (1 + i)^{-t-2}$$

The second-order Taylor approximation is

$$P(i) \approx P(i_0) + P'(i_0)(i - i_0) + \frac{P''(i_0)}{2}(i - i_0)^2$$

The tangent approximation may be rewritten as

$$\frac{P(i) - P(i_0)}{P(i_0)} \approx \frac{P'(i_0)}{P(i_0)}(i - i_0)$$

Using our regression background, for every 100 basis point increase in the interest rate (e.g. from 0.05 to 0.06), the approximate relative price change is  $\frac{P'(i_0)}{P(i_0)}$  percent.

# Modified Duration

The modified duration is defined as

$$ModD = -\frac{P'(i_0)}{P(i_0)} = \frac{\sum_{t \geq 0} C_t t (1+i)^{-t-1}}{\sum_{t \geq 0} C_t (1+i)^{-t}}$$

The larger the modified duration, the more sensitive the price is to interest rate changes.

Macaulay Duration is an adjustment of Modified Duration which is easier to understand heuristically

$$MacD = ModD(1 + i) = \sum_{t \geq 0} \left( \frac{C_t(1 + i)^{-t}}{P(i)} \right) t$$

This is the weighted average of payment times, where the weights are the portion of the total price attributable to that cash flow.

- ① An  $n$ -year zero-coupon bond is purchased to yield  $i$ . Find the Macaulay duration and the modified duration.  $[n, n/(1+i)]$
- ② Find the Macaulay duration of a ten-year 8% 15,000 bond with semiannual coupons and redemption amount 16,500 which yields 3% semiannually.  $[7.411]$
- ③ Find the Macaulay duration of a 15-year mortgage for  $X$  at a monthly nominal interest rate of 6%.  $[6.434]$

## Examples

An  $n$ -year zero-coupon bond is purchased to yield  $i$ . Find the Macaulay duration and the modified duration.  $[n, n/(1 + i)]$

## Examples

Find the Macaulay duration of a ten-year 8% 15,000 bond with semiannual coupons and redemption amount 16,500 which yields 3% semiannually. [7.411]

## Examples

Find the Macaulay duration of a 15-year mortgage for  $X$  at a monthly nominal interest rate of 6%. [6.434]

# First-order Macaulay Approximation

Using the Macaulay duration, we can approximate the change in the present value using the following formula

$$P(i) \approx P(i_0) \left( \frac{1 + i_0}{1 + i} \right)^{MacD}$$

We saw previously that the second-order approximation did a better job than the tangent line. We will define the modified convexity to be:

$$ModC = \frac{P''(i)}{P(i)}$$

The quadratic approximation can then be rewritten as:

$$\begin{aligned}\frac{P(i) - P(i_0)}{P(i_0)} &\approx \frac{P'(i_0)}{P(i_0)}(i - i_0) + \frac{P''(i_0)}{2P(i_0)}(i - i_0)^2 \\ &\approx -ModD(i - i_0) + ModC \frac{(i - i_0)^2}{2}\end{aligned}$$

## Example

A five-year zero-coupon bond redeemable at  $C$  is purchased to provide an annual effective yield of 6%. Find the modified convexity. Use that, along with the modified duration, to estimate the relative price change if the interest rate goes up by 100 basis points.  $[30/(1 + i)^2, -0.04583]$

# Asset-liability Matching Example

Note that for the next few slides we assume a flat yield curve.

You have to pay 1,000 in six months, 1,500 in 12 months, and 2,500 in 18 months. The bonds available for purchase are

- ① Six-month zero-coupon bonds, sold to yield 6% nominal interest convertible semiannually.
- ② 12-month 6% par-value bonds with semiannual coupons.
- ③ 18-month 5% par-value bonds with semiannual coupons.

What face amount of each bond should you purchase to exactly match your liabilities to your assets? [6: 897.11, 12: 1397.11, 18: 2439.02]

## Immunization Example

You have to pay your brother 10,000 in exactly 5 years. He loans you 7,835.26 (to yield him 5%). You can currently buy 5% zero-coupon bonds, but only for either 2, 3, or 7 years. How can you be guaranteed to be able to pay him back?

If you put half (3,917.63) in each type of bond, sell the 7-year bond in three years, and reinvest the entirety in a two-year zero-coupon bond paying the prevailing interest rate, will you have 10,000 in five years if the prevailing interest rate is

- 5%
- 20%
- 1%

Define the surplus  $S(i)$  as the present value of the assets less the present value of the liabilities.

$$S(i) = \sum_{t \geq 0} (A_t - L_t)(1 + i)^{-t} = \sum_{t \geq 0} A_t(1 + i)^{-t} - \sum_{t \geq 0} L_t(1 + i)^{-t}$$

If you can get the following three conditions to hold at your current yield rate

$$S(i) = 0, \quad S'(i) = 0, \quad S''(i) \geq 0$$

then for a small change in  $i$  you will be no worse off. This is Redington immunization.

While the Redington immunization prevents loss for small changes in the interest rate, under full immunization no change in the interest rate will cause a decrease in price. If the following are true:

- $\delta_0$  is the current force of interest (i.e. no yield curve)
- A single liability  $L$  is to be paid at time  $T$
- A pair of assets pay  $U$  and  $W$  at times  $T - u$  and  $T + w$ , respectively ( $0 < u < T$  and  $w > 0$ ).
- The net present value  $S(\delta_0) = 0$  and the derivative  $S'(\delta_0) = 0$ .

Then  $S(\delta) > 0 \quad \forall \quad \delta \neq \delta_0$ .

A liability consists of a series of 15 annual payments of 35,000 with the first payment to be made one year from now.

The assets available to immunize this liability are five-year and ten-year zero-coupon bonds.

The annual effective interest rate used to value the assets and the liability is 6.2%. The liability has the same present value and duration as the asset portfolio.

Calculate the amount invested in the five-year zero-coupon bonds.  
[208,556.21]

An insurance company must pay liabilities of 99 at the end of one year, 102 at the end of two years and 100 at the end of three years. The only investments available to the company are the following three bonds. Bond A and Bond C are annual coupon bonds. Bond B is a zero-coupon bond.

Bond	Maturity (in years)	Yield-to-Maturity (Annualized)	Coupon Rate
A	1	6%	7%
B	2	7%	0%
C	3	9%	5%

All three bonds have a par value of 100 and will be redeemed at par. Calculate the number of units of Bond A that must be purchased to match the liabilities exactly. [0.8807]

Aakash has a liability of 6000 due in four years. This liability will be met with payments of  $A$  in two years and  $B$  in six years. Aakash is employing a full immunization strategy using an annual effective interest rate of 5%.

Calculate  $A$  and  $B$ . [A: 2721.09, B: 3307.50]

Trevor has assets at time 2 of A and at time 9 of B. He has a liability of 95,000 at time 5. Trevor has achieved Redington immunization in his portfolio using an annual effective interest rate of 4%.

Calculate A and B. [A: 48,259.80, B: 47,629.96]

Jia Wen has a liability of 12,000 due in eight years. This liability will be met with payments of 5000 in five years and  $B$  in  $8 + b$  years. Jia Wen is employing a full immunization strategy using an annual effective interest rate of 3%.

Calculate  $b$  and  $B$ . [b: 2.5076, B: 7039.26]