Stat 344

Solutions to Homework Assignment 1

1. Jimmy recently purchased a house for he and his family to live in with a \$300,000 30-year mortgage. He is worried that should he die before the mortgage is paid, his family will not be able to afford the mortgage payment. A friend of his suggests that he purchase a life insurance policy.

   (a) What type of life insurance product, if any, should Jimmy purchase? (Your answer should be one to three sentences in length.)

   (b) Before being issued the policy, Jimmy is required to go through an underwiting procedure. In your own words, briefly describe what is meant by the term **underwriting** and what it might entail for Jimmy and the insurance company. (Your answer should be two to four sentences in length.)

   Answer:

   (a) Jimmy should consider purchasing a **term life insurance product**. This type of policy will offer death benefit coverage for a set period of time; 30 years would be sensible inthis case to match the length of his mortgage. He might consider a policy with a decreasing death benefit, since his mortgage will be decreasing in value; though term policies with a level face amount are much more common in the U.S.

   (b) Underwriting is the process by which a life insurance company collects information about a prospective insured being issuing a life insurance policy. They do this to make sure that the insured in indeed insurable (i.e., isn't facing imminent death or having a terminal illness) as well as to help make an assessment of the riskiness of the individual. This then allows the insurer to charge an appropriate rate for the policy (this is usually done by assigning the insured to a particular risk class).

2. Consider a proposed survival function $S_0(t) = \dfrac{1}{10}\sqrt{100-t}, \quad 0 \leq t \leq 100.$

   (a) Verify that this is indeed a valid survival function. (That is, verify that it meets the three necessary conditions discussed in class.)

   Answer:

   i. $S_0(0) = \frac{1}{10}\sqrt{100-0} = \frac{1}{10} \cdot 10 = 1.$

   ii. $S_0(100) = \frac{1}{10}\sqrt{100-100} = 0$, so all lives die by age 100.

   iii. $\frac{d}{dt}S_0(t) = \frac{d}{dt}\frac{1}{10}\sqrt{100-t} = \frac{1}{10} \cdot \left(-\frac{1}{2}\right)(100-t)^{-1/2} \leq 0 \; \forall t \in [0,100].$

Thus all of the required conditions are met.

(b) Find the probability that a newborn dies between the ages of 10 and 20.

Answer: $S_0(10) - S_0(20) = \frac{1}{10}\sqrt{100-10} - \frac{1}{10}\sqrt{100-20} = 0.949 - 0.894 = 0.054.$

(c) Find the probability that a 30-year old lives to at least age 60.

Answer: Here we must find the probability that a newborn lives to age 60, *conditioned* on the fact that they lived to at least age 30.

$$Pr\left[T_0 > 60 \,|\, T_0 > 30\right] = \frac{S_0(60)}{S_0(30)} = \frac{0.1\sqrt{100-60}}{0.1\sqrt{100-30}} = \sqrt{\frac{40}{70}} = 0.756.$$

We could also answer this question by finding the survival function for a 30-year old, $S_{30}(t)$.

(d) Find the median lifetime length for a newborn.

Answer: Here we want to find the age beyond which the probability of survival is 0.5, that is, the age t at which $S_0(t) = 0.5$.

$$\begin{aligned} 0.5 &= \frac{1}{10}\sqrt{100-t} \\ 5 &= \sqrt{100-t} \\ 25 &= 100 - t \\ t &= 75 \end{aligned}$$

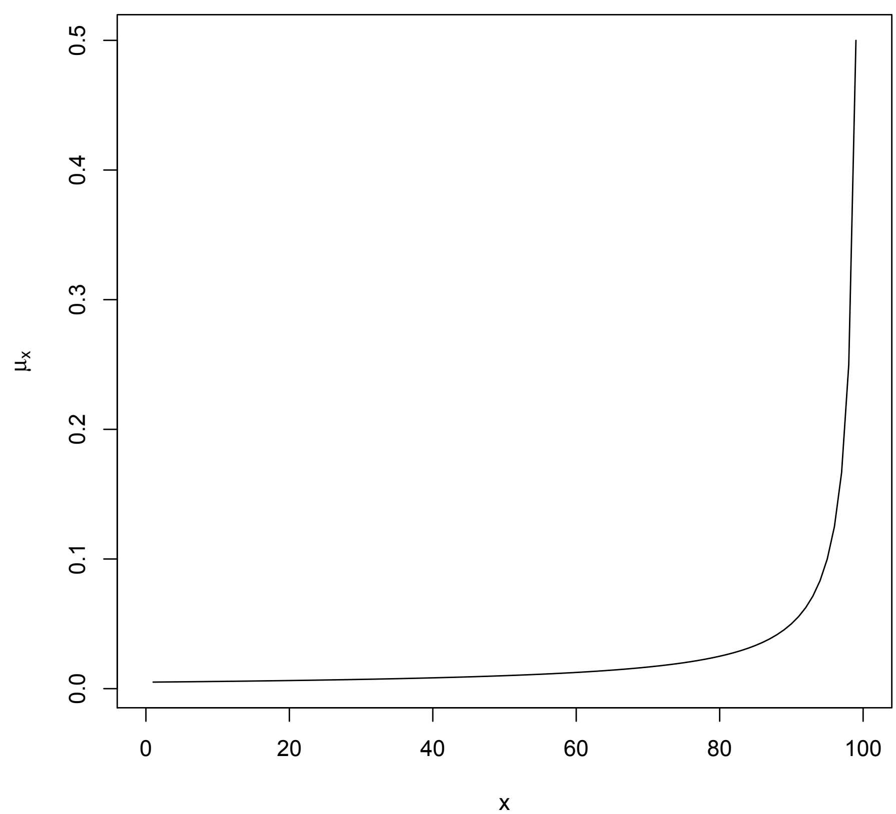
(e) Find an expression for the force of mortality, μ_x, simplifying as far as possible.

Answer:

$$\begin{aligned} \mu_x &= \frac{-\frac{d}{dx}S_0(x)}{S_0(x)} \\ &= \frac{-\frac{1}{10}\cdot\left(-\frac{1}{2}\right)(100-x)^{-1/2}}{\frac{1}{10}\sqrt{100-x}} \\ &= \frac{1}{2(100-x)} \end{aligned}$$

(f) Sketch the force of mortality μ_x for $x \in [0, 100]$.

Answer:



(g) Find the mean of the random variable T_{30}.

Answer: First, we calculate ${}_tp_{30} = \dfrac{S_0(30+t)}{S_0(30)} = \dfrac{1}{\sqrt{70}}\,(70-t)^{1/2}$. Then

$$\begin{aligned} E[T_{30}] &= \int_0^{70} {}_tp_{30}\,dt \\ &= \int_0^{70} \frac{1}{\sqrt{70}}\,(70-t)^{1/2}\,dt \\ &= 46.67. \end{aligned}$$

(h) Find the variance of the random variable T_{30}.

Answer:

$$\begin{aligned} E[T_{30}^2] &= 2\int_0^{70} t\ {}_tp_{30}\,dt \\ &= \frac{2}{\sqrt{70}}\int_0^{70} t\ (70-t)^{1/2}\ dt \\ &= 2613.333, \end{aligned}$$

after applying integration by parts. Then $Var(T_{30}) = E[T_{30}^2] - E[T_{30}]^2 = 435.5549$.

(i) Calculate $\mathring{e}_{30:\overline{20|}}$.

Answer:

$$\begin{aligned} \mathring{e}_{30:\overline{20|}} &= \int_0^{20} \frac{1}{\sqrt{70}}\,(70-t)^{1/2}\ dt \\ &= \frac{-2}{3\sqrt{70}}(70-t)^{3/2}\Big|_0^{20} \\ &= 18.49 \end{aligned}$$

(j) Find the median future lifetime of a person age 30. (Is your answer consistent with your answer to part (c)?)

Answer:

$$\begin{aligned} S_{30}(t) &= \frac{S_0(30+t)}{S_0(30)} \\ &= \frac{1}{\sqrt{70}}\,(70-t)^{1/2} \end{aligned}$$

Then

$$\frac{1}{\sqrt{70}}\,(70-t)^{1/2} = 0.5$$

so that $t = 52.5$ is the median future lifetime of a person age 30. In part (c), the probability of living at least 30 more years was greater than 0.5, which is consistent with this result.

3. Assume that the force of mortality for a survival model is given by $\mu_x = \dfrac{1}{110-x}$.

(a) Find the survival function $S_0(t)$ corresponding to this force of mortality, simplifying as far as possible.

Answer:

$$
\begin{aligned}
S_0(t) &= \exp\left\{-\int_0^t \mu_{0+s}\,ds\right\} \\
&= \exp\left\{-\int_0^t \frac{1}{110-s}\,ds\right\} \\
&= \exp\left\{\ln(110-s)|_0^t\right\} \\
&= \exp\left\{\ln(110-t) - \ln(110)\right\} \\
&= \exp\left\{\ln\left(\frac{110-t}{110}\right)\right\} \\
&= \frac{110-t}{110} \\
&= 1 - \frac{t}{110}
\end{aligned}
$$

(b) What is the limiting age ω for this model?

Answer: $S_0(t) = 0$ for $t = 110$ so that $\omega = 110$.

(c) Sketch the survival function $S_0(t)$.

Answer: (See figure.)

(d) Find the density function $f_0(t)$.

Answer:

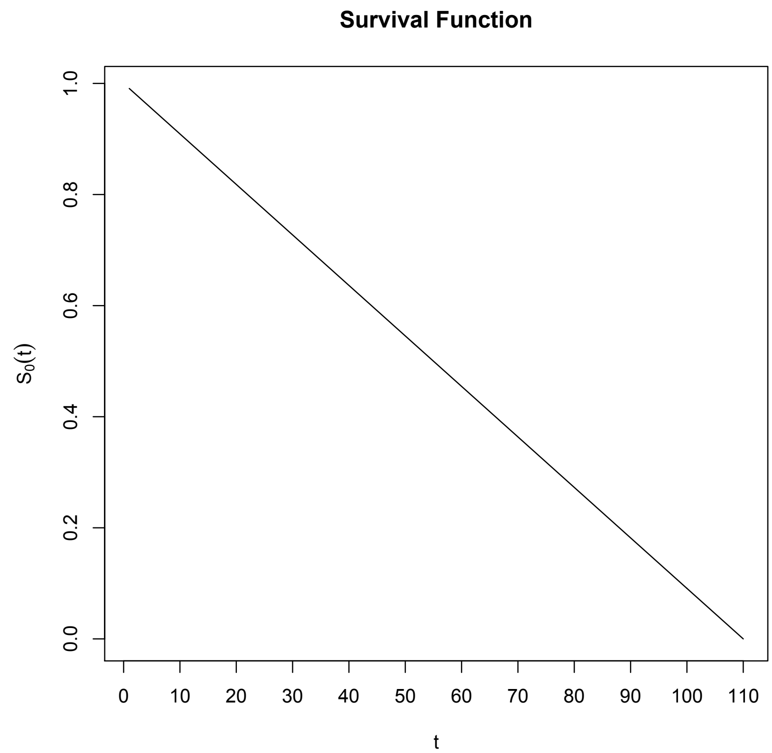
$$
\begin{aligned}
f_0(t) &= -\frac{d}{dt}S_0(t) \\
&= -\frac{d}{dt}\left[1 - \frac{t}{110}\right] \\
&= \frac{1}{110}
\end{aligned}
$$

(e) Calculate the probability of a newborn dying between the ages of 40 and 60.

Answer:

$$
\begin{aligned}
S_0(40) - S_0(60) &= \left(1 - \frac{40}{110}\right) - \left(1 - \frac{60}{110}\right) \\
&= \frac{20}{110}
\end{aligned}
$$

(f) Calculate the probability of a 20-year-old dying between the ages of 40 and 60.



Answer: First, we find that

$$S_{20}(t) = \frac{S_0(20+t)}{S_0(20)} = 1 - \frac{t}{90}$$

Then

$$\begin{aligned} S_{20}(20) - S_{20}(40) &= \left(1 - \frac{20}{90}\right) - \left(1 - \frac{40}{90}\right) \\ &= \frac{20}{90} \end{aligned}$$

(g) Do you think this survival model is suitable as a model for human mortality? Why or why not? (Your answer should be one or two sentences in length.)

Answer: We can see that the survival function is linearly decreasing and that the density function is a constant. That is, the density for the lifetime random variable is NOT a function of age. This property seems like it would not be a very realistic representation of human mortality patterns. Also, in this model, survival beyond age 110 is not possible; this is unrealistic.

4. You are given the following information:

$$_3p_{51} = 0.9126, \qquad {}_2q_{50} = 0.0298, \qquad q_{52} = 0.0300, \qquad {}_2p_{52} = 0.9312, \qquad q_{54} = 1$$

(a) Find the numerical values of the following quantities (and also make sure you understand the interpretation of each):

i. p_{52}

Answer: $p_{52} = 1 - q_{52} = 1 - 0.0300 = 0.9700.$

ii. $_4p_{50}$

Answer:

$$\begin{aligned} {}_4p_{50} = {}_2p_{50}\,{}_2p_{52} &= (1 - {}_2q_{50})({}_2p_{52}) \\ &= (1 - 0.0298)(0.9312) \\ &= 0.9035 \end{aligned}$$

iii. p_{51}

Answer: $p_{51}\,{}_2p_{52} = {}_3p_{51}$ so that

$$\begin{aligned} p_{51} &= \frac{{}_3p_{51}}{{}_2p_{52}} \\ &= \frac{0.9126}{0.9312} \\ &= 0.9800 \end{aligned}$$

iv. $_{2|2}q_{50}$

Answer:

$$\begin{aligned} {}_{2|2}q_{50} &= {}_2p_{50}\,{}_2q_{52} \\ &= (1 - {}_2q_{50})(1 - {}_2p_{52}) \\ &= (1 - 0.0298)(1 - 0.9312) \\ &= 0.0667 \end{aligned}$$

v. $_{3|}q_{50}$

Answer: $_{3|}q_{50} = {}_3p_{50}\,q_{53} = ({}_2p_{50}\,p_{52})(q_{53})$

Now in order to find q_{53} we note that

$$p_{52}\,p_{53} = {}_2p_{52}$$

so that

$$p_{53} = {}_2p_{52}\,/\,p_{53} = 0.9312/0.9700 = 0.9600.$$

Then $q_{53} = 1 - p_{53} = 1 - 0.9600 = 0.0400$. Finally,

$$\begin{aligned} {}_{3|}q_{50} &= ({}_2p_{50}\, p_{52})(q_{53}) \\ &= (1 - {}_2p_{50})(p_{52})(q_{53}) \\ &= (1 - 0.0298)(0.97)(1 - 0.9600) \\ &= 0.0376. \end{aligned}$$

vi. ${}_2p_{53}$

Answer:

$${}_2p_{53} = p_{53}p_{54} = p_{53}(1 - q_{54}) = (0.96)(1 - 1) = 0$$

vii. e_{50}

Answer:

$$\begin{aligned} e_{50} &= \sum_{k=1}^{\infty} {}_kp_{50} = {}_1p_{50} + {}_2p_{50} + {}_3p_{50} + {}_4p_{50} + {}_5p_{50} + \cdots \\ &= 0.99 + (0.99)(0.98) + (0.99)(0.98)(0.97) + (0.99)(0.98)(0.97)(0.96) + 0 + 0 + \cdots \\ &= 3.8047 \end{aligned}$$

(b) For the random variable K_{51}, find its:

i. pmf (probability mass function)

Answer: $P[K_{51} = k] = {}_{k+1}p_{51} - {}_kp_{51}$

k	$\mathbf{P[K_{51} = k]}$
0	0.02
1	0.0294
2	0.038
3	0.9126
4+	0

ii. mean

Answer: $e_{51} = 2.8432$

iii. standard deviation

Answer:

$$E[K_{51}^2] = 0^2(0.02) + 1^2(0.0294) + 2^2(0.038) + 3^2(0.9126) + 4^2(0) + \cdots = 8.3948$$

so that the standard deviation is

$$\sqrt{8.3948 - (2.8432)^2} = 0.5577$$

iv. mode

Answer: The mode for K_{51} is 3.

v. Calculate the probability that K_{51} has an odd value.

Answer: $0.0294 + 0.9126 = 0.942$

5. Show that $\frac{d}{dx}\,{}_tp_x = {}_tp_x\,(\mu_x - \mu_{x+t})$.

Answer:

$$
\begin{aligned}
\frac{d}{dx}\,{}_tp_x &= \frac{d}{dx}\,S_x(t) \\
&= \frac{d}{dx}\,\frac{S_0(x+t)}{S_0(x)} \\
&= \frac{d}{dx}\,S_0(x+t)[S_0(x)]^{-1} \\
&= -S_0(x+t)[S_0(x)]^{-2}S_0'(x) + [S_0(x)]^{-1}S_0'(x+t) \\
&= \frac{-S_0'(x)}{S_0(x)} \cdot \frac{S_0(x+t)}{S_0(x)} + \frac{S_0'(x+t)}{S_0(x)} \\
&= \mu_x\,{}_tp_x + \frac{S_0'(x+t)}{S_0(x)} \\
&= \mu_x\,{}_tp_x + \frac{S_0'(x+t)}{S_0(x+t)} \cdot \frac{S_0(x+t)}{S_0(x)} \\
&= \mu_x\,{}_tp_x - \frac{-S_0'(x+t)}{S_0(x+t)} \cdot \frac{S_0(x+t)}{S_0(x)} \\
&= \mu_x\,{}_tp_x - \mu_{x+t}\,{}_tp_x \\
&= {}_tp_x\,(\mu_x - \mu_{x+t})
\end{aligned}
$$