Stat 344 Homework Assignment 4 Solutions

General Notes:

- Please hand in Part I on paper in class on the due date.
- Also submit on Learning Suite the Excel spreadsheet you create to answer the questions in Part II. Your spreadsheet should be neatly organized and labeled; each answer should be highlighted in some manner, and it should be very clear how each of your answers was obtained.
- If you use **R**, submit the code you use, as well as your output. Your document should be neatly organized and labeled; it should be very clear how each of your answers was obtained.

Part I

1. You are given the following life table excerpt (Table 3.1 from Dickson et al.). Assume $i = 8\%$.

x	ℓ_x	d_x
30	$10,000.00$	34.78
31	$9,965.22$	38.10
32	$9,927.12$	41.76
33	$9,885.35$	45.81
34	$9,839.55$	50.26
$\vdots$	$\vdots$	$\vdots$

(a) Calculate the mean and variance of the present value of the benefit for a 2-year term life annuity due issued to (30). [1.9227, 0.00293]

Answer: To get the expected value, we can calculate:

$$\ddot{a}_{30:\overline{2}|} = 1 + vp_{30} = 1.9227$$

For the variance, we first can calculate

$$A_{30:\overline{2}|} = A^{1}_{30:\overline{2}|} + A_{30:\overset{\;1}{\overline{2}|}} = vq_{30} + v^2 p_{30} q_{31} + v^2 {}_2p_{30} = 0.8576$$

and

$$ {}^2A_{30:\overline{2}|} = \left(v^2\right) q_{30} + \left(v^2\right)^2 p_{30} q_{31} + \left(v^2\right)^2 {}_2p_{30} = 0.7355 $$

Then the variance is

$$ \frac{{}^2A_{30:\overline{2}|} - \left(A_{30:\overline{2}|}\right)^2}{d^2} = 0.00293 $$

(b) Calculate $\ddot{a}^{(2)}_{30:\overline{2}|}$ using the following assumptions. (Keep at least 4 decimal places and please show all work.)

(i) Woolhouse's approximation with two terms [1.885473]

(ii) Woolhouse's approximation with three terms using $\mu_x \overset{CF}{=} -\frac{\log(p_{x-1})+\log(p_x)}{2}$ and $p_{29} = 0.997$. [1.884767]

(iii) UDD [1.88472]

(iv) CFM [1.88473]

Answer:

$$ p_{30} = 0.996522 \text{ and } p_{31} = 0.9961767 $$

(i) $\ddot{a}^{(2)}_{30:\overline{2}|} \approx \ddot{a}_{30:\overline{2}|} - \frac{1}{4}\left(1 - v^2\, {}_2p_{30}\right) = 1.9227 - \frac{1}{4}\left(1 - (1/1.08)^2\,(0.992712)\right) = \boxed{1.885473}$

(ii)

$$ \ddot{a}^{(2)}_{30:\overline{2}|} \approx \ddot{a}_{30:\overline{2}|} - \frac{1}{4}\left(1 - v^2\, {}_2p_{30}\right) - \frac{3}{48}\left(\ln(1.08) + 0.00324 - v^2\, {}_2p_x(\ln(1.08) + 0.00402)\right) $$
$$ = \boxed{1.884767} $$

(iii) $i^{(2)} = 0.07846097$ and $d^{(2)} = 0.0754991$ and $d = 0.07407407$ so that $\alpha(2) = 1.00037$ and $\beta(2) = 0.2598076$. Then

$$ \ddot{a}^{(2)}_{30:\overline{2}|} \approx \alpha(2) \cdot \ddot{a}_{30:\overline{2}|} - \beta(2)\left(1 - v^2\, {}_2p_{30}\right) = \boxed{1.88472} $$

(iv) $\ddot{a}^{(2)}_{30:\overline{2}|} \approx \frac{1}{2}(1 + v^{0.5}\, {}_{0.5}p_{30} + v\, p_{30} + v^{1.5}\, {}_{1.5}p_{30}) \overset{CF}{=} \boxed{1.88473}$

2. Jimbo is age 65 and is about to retire and has saved $\$1,000,000$ for his retirement. Assume that his mortality is given by the Standard Ultimate Mortality Model. He anticipates needing about $\$50,000$ in his first year of retirement to cover his expenses. His expenses would increase with inflation (e.g., 1% per year) in subsequent years. He is considering the following options:

   (i) Invest the $\$1,000,000$ in a bank account earning 3% interest. He would make withdrawals from the bank account to pay his expenses.

   (ii) Use the $\$1,000,000$ to purchase a whole life annuity making level payments of X per year.

(iii) Invest the \$1,000,000 in AA or A 30-year corporate bonds paying annual 6% coupons.

(a) Assuming $i = 5\%$, calculate X. [73801.88]

(b) List the advantages and disadvantages (or risks) associated with each option.

(c) Recommend with reasons a strategy that you believe would be superior to the options listed above.

Answer:

(a) $1,000,000 = X\ddot{a}_{65}$ so that $X = 73,801.88$.

(b) (i) Advantages:
- Little or no chance of losing principle
- Get to maintain principle (and control thereof) for possible future opportunities; investment is extremely liquid

Disadvantages:
- Will have to start eating into principle soon to fund expenses; could run out of funds prior to dying (i.e., there's significant longevity risk)
- 3% is a low return on investment

(ii) Advantages:
- Eliminates longevity risk
- Eliminates investment risk (interest rate risk)

Disadvantages:
- Annual payments will be constant, whereas expenses will be increasing. Payments will be more than adequate at first, but may end up being inadequate later, especially if inflation is greater than expected.
- Lose control of capital
- Could get a very poor return if Jimbo dies soon

(iii) Advantages:
- Rate of return is better than the bank account
- Could sell bonds if necessary; investment is fairly liquid

Disadvantages:
- These investments have default risk (and also interest rate risk and some liquidity risk)
- Lose control of capital
- Coupon payments will be constant, whereas expenses will be increasing

(c) Purchase a geometrically increasing life-and-10-year-certain annuity-due. First payment would be $\$50,000$ and would increase by 1% per year to match inflation. This eliminates longevity risk; also the 10 year guarantee eliminates the chance of a very poor return. This annuity would cost about $\$761,000$ under the assumptions given. The remainder of the money could be deposited into a bank account (or other liquid investment), which would provide liquidity in case of unexpected expenses or another investment opportunity. (Or if inflation is more than anticipated, which is a risk that still remains, this money could make up the difference between the annuity payment and the actual expenses.)

3. Let Y denote the present value of the benefit for a whole life annuity paying continuously at a rate of $\$10,000$ per year, issued to (50).

(a) Write an expression for Y in terms of the future lifetime random variable T_{50}.

Answer:

$$Y = 10,000\, \bar{a}_{\overline{T_{50}}|} = 10,000\left(\frac{1-e^{-\delta\, T_{50}}}{\delta}\right) = 10,000\left(\frac{1-v^{T_{50}}}{\delta}\right)$$

Now suppose you are given that:

$$\bar{A}_{50} = \frac{(1-e^{-40\,\delta})}{32\,\delta}, \qquad \delta = 0.06$$

(b) Find the expected value of Y. [87735.93]

Answer:

$$\begin{aligned} E[Y] &= 10,000\bar{a}_{50} \\ &= 10,000\left[\frac{1-\bar{A}_{50}}{\delta}\right] \\ &= 10,000\left[\frac{1-\frac{(1-e^{-40\,\delta})}{32\,\delta}}{\delta}\right] \\ &= 10,000\left[\frac{1-\frac{(1-e^{-40\,(0.06)})}{32\,(0.06)}}{0.06}\right] \\ &= 10,000\left[\frac{1-0.47358}{0.06}\right] \\ &= \boxed{87{,}735.93} \end{aligned}$$

(c) Find the standard deviation of Y. [30728.19]

Answer: First, we calculate

$$\begin{aligned}
{}^{2}\bar{A}_{50} &= \left[\frac{(1-e^{-40\,(0.12)})}{32\,(0.12)}\right] \\
&= 0.25827
\end{aligned}$$

Then

$$\begin{aligned}
Var(Y) &= (10,000)^2 \left[\frac{{}^{2}\bar{A}_{50} - \left(\bar{A}_{50}\right)^2}{\delta^2}\right] \\
&= (10,000)^2 \left[\frac{0.25827 - (0.47358)^2}{(0.06)^2}\right] \\
&= 944,221,766.7
\end{aligned}$$

so that the standard deviation of Y is $\boxed{30{,}728.19}$.

4. You are given:

$$\ddot{a}_{60} = 7.242, \quad A_{\overset{1}{40}:\overline{10|}} = 0.35, \quad i = 10\%, \quad {}_{10}p_{50} = 0.8$$

(a) Describe in words what is meant by the symbol $\ddot{a}_{\overline{40:\overline{20|}}}$.

Answer: This is the EPV of a whole life annuity due issued to (40), with the first 20 payments guaranteed.

(b) Calculate $\ddot{a}_{\overline{40:\overline{20|}}}$. [10.14671]

Answer:

$$\begin{aligned}
\ddot{a}_{\overline{40:\overline{20|}}} &= \ddot{a}_{\overline{20|}} + {}_{20}E_{40}\,\ddot{a}_{60} \\
&= \ddot{a}_{\overline{20|}} + {}_{10}E_{40}\;{}_{10}E_{50}\,\ddot{a}_{60} \\
&= \ddot{a}_{\overline{20|}} + A_{\overset{1}{40}:\overline{10|}}\;{}_{10}p_{50}\,v^{10}\,\ddot{a}_{60} \\
&= 9.36492 + (0.35)(0.8)\left(\frac{1}{1.1}\right)^{10}(7.242) \\
&= \boxed{10.14671}
\end{aligned}$$

5. Suppose that mortality for (x) is given by the DeMoivre model with limiting age ω and the force of interest is δ.

(a) Show that

$$\bar{a}_x = \frac{(\omega - x) - \bar{a}_{\overline{\omega-x}|}}{\delta(\omega - x)}$$

Suppose that $x = 40, \omega = 100$ and $\delta = 0.1$. Let Y be the PV of benefits RV for a continuous whole life annuity issued to (x).

(b) Calculate $\bar{a}_x$. [8.337465]

(c) Calculate $P(Y > \bar{a}_x)$. [0.7009597]

Answer:

(a)

$$\begin{aligned}
\overline{a}_x &= \int_0^{\omega-x} \overline{a}_{\overline{t}|}\, {}_tp_x\, \mu_{x+t}\, dt \\
&= \int_0^{\omega-x} \frac{1-e^{-\delta\, t}}{\delta} \cdot \frac{1}{\omega - x}\, dt \\
&= \frac{1}{\delta(\omega - x)} \int_0^{\omega-x} 1 - e^{-\delta\, t}\, dt \\
&= \frac{1}{\delta(\omega - x)} \left[t + \frac{e^{-\delta\, t}}{\delta} \right]_0^{\omega-x} \\
&= \frac{1}{\delta(\omega - x)} \left(\left[(\omega - x) + \frac{e^{-\delta\,(\omega-x)}}{\delta} \right] - \left[0 + \frac{1}{\delta} \right] \right) \\
&= \frac{1}{\delta(\omega - x)} \left((\omega - x) + \frac{e^{-\delta\,(\omega-x)}}{\delta} - \frac{1}{\delta} \right) \\
&= \frac{1}{\delta(\omega - x)} \left((\omega - x) - \left[\frac{1 - e^{-\delta\,(\omega-x)}}{\delta} \right] \right) \\
&= \frac{(\omega - x) - \overline{a}_{\overline{\omega-x}|}}{\delta(\omega - x)}
\end{aligned}$$

(b) $\bar{a}_x = \frac{(\omega-x)-\bar{a}_{\overline{\omega-x}|}}{\delta(\omega-x)} = \boxed{8.337465}$

(c)

$$\begin{aligned}
P(Y > \bar{a}_x) &= P(Y > 8.337465) \\
&= P\left(\frac{1-\exp(-\delta\, T_x)}{\delta} > 8.337465\right) \\
&= P\left(1-\exp(-\delta\, T_x) > 0.8337465\right) \\
&= P(\exp(-\delta\, T_x) < 0.1662535) \\
&= P(\delta\, T_x > 1.794242) \\
&= P(T_x > 17.94242) \\
&= 1 - 17.94242/60 = \boxed{0.7009597}
\end{aligned}$$

6. Whole life annuities-due are issued to 100 independent lives age 50. Each annuity pays 20 per year. Mortality is given by the Standard Ultimate Survival Model, and $i = 5\%$. For convenience, the following values are provided (you could/should verify these values from the spreadsheet):

$$A_{50} = 0.189308 \qquad {}^2A_{50} = 0.051075 \qquad \ddot{a}_{50} = 17.0245 \qquad d = 0.0476$$

   (a) Using a normal approximation, estimate the probability that the total PV of payments will exceed 35,000. [0.03336]

   (b) Now suppose that there is a correlation of 0.2 between each pair of lives. Explain in words (without calculation) how and why your answer to the above problem would change. It may help to draw a picture. (**BONUS:** Calculate this value using a normal approximation.) [0.3438]

   (c) Now suppose that the lives are perfectly correlated, i.e., all 100 lives will have identical future lifetimes.

      (i) Calculate the probability that the total PV of payments will exceed 35,000. (Hint: You'll need additional values from the spreadsheet for this part.) [0.58489]

      (ii) Repeat the previous part using a normal approximation. [0.4272584]

Answer:

(a) Let Y_i be the PV of payments for individual i ($i = 1, 2, \ldots, 100$). Then

$$E[Y_i] = 20\ddot{a}_{50} = 340.49 \text{ and } Var[Y_i] = (20)^2\left(\frac{{}^2A_{50} - A_{50}^2}{d^2}\right) = 2,690.043$$

Letting Y^{tot} denote the total PV of payments,

$$E\left[Y^{tot}\right] = 100 \cdot E[Y_i] = 34,049 \text{ and } Var\left[Y^{tot}\right] = 100 \cdot Var[Y_i] = 269,004$$

Then

$$\begin{aligned}
P(Y^{tot} > 35,000) &= P\left(\frac{Y^{tot} - E\left[Y^{tot}\right]}{\sqrt{Var\left[Y^{tot}\right]}} > \frac{35,000 - E\left[Y^{tot}\right]}{\sqrt{Var\left[Y^{tot}\right]}}\right) \\
&= P\left(Z > \frac{35,000 - E\left[Y^{tot}\right]}{\sqrt{Var\left[Y^{tot}\right]}}\right) \\
&= P\left(Z > \frac{35,000 - 34,049}{\sqrt{269,004}}\right) \\
&= P(Z > 1.83) \\
&= \boxed{0.03336}
\end{aligned}$$

(b) In this case, the variance of the total PV of payments will be increased. As a result, the probability that the PV of payments will exceed 35000 will also increase.

The variance of Y^{tot} in this case is

$$\begin{aligned}
Var(Y^{tot}) &= 100Var(Y_i) + 2\binom{100}{2}Cov(Y_i, Y_j) \\
&= 100(2690.043) + 2(4950)(0.2)(2690.043) \\
&= 5,595,289
\end{aligned}$$

so that

$$\begin{aligned}
P(Y^{tot} > 35,000) &= P\left(\frac{Y^{tot} - E\left[Y^{tot}\right]}{\sqrt{Var\left[Y^{tot}\right]}} > \frac{35,000 - E\left[Y^{tot}\right]}{\sqrt{Var\left[Y^{tot}\right]}}\right) \\
&= P\left(Z > \frac{35,000 - E\left[Y^{tot}\right]}{\sqrt{Var\left[Y^{tot}\right]}}\right) \\
&= P\left(Z > \frac{35,000 - 34,049}{\sqrt{5,595,289}}\right) \\
&= P(Z > 0.40) \\
&= \boxed{0.3438}
\end{aligned}$$

(c) (i) $2000\ddot{a}_{\overline{37}|} > 35000$ and $2000\ddot{a}_{\overline{36}|} < 35000$, so the lives have to live at least 36 years. The probability of this happening is ${}_{36}p_{50} = \boxed{0.58489}$

(ii) In this case, $E[Y^{tot}] = (100)(20)\ddot{a}_{50} = 34,049$ and $Var(Y^{tot}) = (2000)^2\left(\frac{{}^2A_{50} - A_{50}^2}{d^2}\right) =$

$26,900,430$ so that

$$
\begin{aligned}
P(Y^{tot} > 35,000) &= P\left(\frac{Y^{tot} - E\left[Y^{tot}\right]}{\sqrt{Var\left[Y^{tot}\right]}} > \frac{35,000 - E\left[Y^{tot}\right]}{\sqrt{Var\left[Y^{tot}\right]}}\right) \\
&= P\left(Z > \frac{35,000 - E\left[Y^{tot}\right]}{\sqrt{Var\left[Y^{tot}\right]}}\right) \\
&= P\left(Z > \frac{35,000 - 34,049}{\sqrt{26,900,430}}\right) \\
&= P(Z > 0.18) \\
&= \boxed{0.4272584}
\end{aligned}
$$

7. A 30-year deferred whole life annuity-due is issued to (35), paying an amount of 1 per year, with the extra feature that the net single premium is refunded (without any accumulated interest) at the end of the year of death if (35) dies during the deferral period. Calculate the net single premium for this product, given the following values: [1.490323]

$$\ddot{a}_{65} = 9.90 \qquad A_{35:\overline{30|}} = 0.21 \qquad A^{1}_{35:\overline{30|}} = 0.07$$

Answer:

$$
\begin{aligned}
P &= PA^{1}_{35:\overline{30|}} + {}_{30}E_{35}\,\ddot{a}_{65} \\
P &= P(0.07) + (0.14)(9.9) \\
P &= \frac{(0.14)(9.9)}{0.93} \\
P &= 1.490323
\end{aligned}
$$

8. Consider a continuous whole life annuity (paying at a rate of \$1 per year) issued to a person age x. Letting Y denote the PV of this annuity benefit, we have shown that we can write the PV of this benefit as $Y = \bar{a}_{\overline{T_x|}}$ where T_x is the future lifetime random variable for (x). Consider the cumulative distribution function (CDF) for Y, i.e., $F_Y(y) = Pr(Y \leq y)$.

   (a) What are the bounds on the values of Y? Explain the limiting values of T_x they correspond to.

   (b) Show that

   $$F_Y(y) = F_x\left(\frac{-\ln(1-\delta\, y)}{\delta}\right) \qquad \text{for} \quad b_1 < y < b_2$$

   where F_x is the cumulative distribution function for T_x and b_1 and b_2 are the bounds from the previous part.

(c) Show that we can write $P(u \leq Y \leq w)$ as ${}_{b|c}q_x$, where b and c are values you should specify. (Just find the correct values of b and c; no further proof is required.)

Answer:

(a) If the individual dies immediate after purchasing the annuity, the T_x will be 0 and the PV of the annuity benefit (Y) will also be 0. In the other extreme, as the person's future lifetime tends to infinity ($T_x \to \infty$), their life annuity will approach a perpetuity. In this case Y will approach the corresponding perpetuity value, $1/\delta$. In no case can this annuity have a PV that exceeds $1/\delta$, the PV of a continuous perpetuity.

(b)

$$\begin{aligned} F_Y(y) &= Pr[Y \leq y] \\ &= Pr\left[\bar{a}_{\overline{T_x}|} \leq y\right] \\ &= Pr\left[\frac{1 - v^{T_x}}{\delta} \leq y\right] \\ &= Pr\left[1 - v^{T_x} \leq \delta\, y\right] \\ &= Pr\left[-v^{T_x} \leq \delta\, y - 1\right] \\ &= Pr\left[v^{T_x} \geq 1 - \delta\, y\right] \\ &= Pr\left[e^{-\delta\, T_x} \geq 1 - \delta\, y\right] \\ &= Pr\left[-\delta\, T_x \geq \ln(1 - \delta\, y)\right] \\ &= Pr\left[T_x \leq \frac{-\ln(1 - \delta\, y)}{\delta}\right] \\ &= F_x\left(\frac{-\ln(1 - \delta\, y)}{\delta}\right) \end{aligned}$$

(c)

$$\begin{aligned} P(u \leq Y \leq w) &= F_Y(w) - F_Y(u) \\ &= F_x\left(\frac{-\ln(1 - \delta\, w)}{\delta}\right) - F_x\left(\frac{-\ln(1 - \delta\, u)}{\delta}\right) \\ &= P\left(\frac{-\ln(1 - \delta\, u)}{\delta} \leq T_x \leq \frac{-\ln(1 - \delta\, w)}{\delta}\right) \\ &= {}_{b|c}q_x, \end{aligned}$$

where $b = \frac{-\ln(1-\delta\, u)}{\delta}$ and $c = \frac{-\ln(1-\delta\, w)}{\delta} - \frac{-\ln(1-\delta\, u)}{\delta}$

Part II

For this part, we will assume that mortality is given by the Standard Ultimate Mortality Model (typically, annuities aren't underwritten, so we're using ultimate mortality rather than select mortlity for this assignment) and the interest rate is a level 5% annual effective rate. As was the case for the first assignment y is the last two digits of your BYU ID number; if your BYU ID number is between $00 - 19$, add 20.

Deterministic Analysis

Subpart A: Level Payment Whole Life Annuity

Let Y_1 denote the present value of a whole life annuity due, issued to (y), with level annual payments of $\$60,000$ per year.

1. Find $E[Y_1]$.
2. Find the standard deviation of Y_1.
3. Find $Pr[Y_1 < \$500,000]$.
4. Create a graph showing the probability mass function of Y_1.

Subpart B: Level Payment Guaranteed Annuity

Let Y_2 denote the present value of a life-and-10-year-certain annuity due, issued to (y), with level annual payments of $\$60,000$ per year.

1. Find $E[Y_2]$.
2. Find the standard deviation of Y_2.
3. Find $Pr[Y_2 < \$500,000]$.
4. Create a graph showing the probability mass function of Y_2.

Subpart C: Geometrically Increasing Whole Life Annuity

Let Y_3 denote the present value of a whole life annuity due, issued to (y), with a first payment of $\$50,000$, and each subsequent payment increasing by 2%.

1. Find $E[Y_3]$.
2. Find the standard deviation of Y_3.

3. Find $Pr[Y_3 < \$500,000]$.

4. Create a graph showing the probability mass function of Y_3.

Subpart D: Geometrically Increasing Guaranteed Annuity

Let Y_4 denote the present value of a life-and-10-year-certain annuity due, issued to (y), with a first payment of \$50,000, and each subsequent payment increasing by 2%.

1. Find $E[Y_4]$.

2. Find the standard deviation of Y_4.

3. Find $Pr[Y_4 < \$500,000]$.

4. Create a graph showing the probability mass function of Y_4.

Optional Bonus Section – worth up to 10% bonus

Note: This part is perhaps easier to complete in **R** than Excel. You can use **R** for this part, even if you used Excel for the deterministic analysis.

Stochastic Analysis

Now you will consider an analysis based on simulations. Specifically, you will simulate the future lifetime of the same y-year old 1,000 times and base your answers on the results of these simulations. Use the same mortality and interest assumptions as above. (Note: You can use 10,000 or 100,000 simulations if you prefer and if it doesn't cause a severe processing burden for your computer.)

Subpart A: Level Payment Whole Life Annuity

Redo Questions 1-4 in Part II — Subpart A based on your simulated values.

Subpart B: Level Payment Guaranteed Annuity

Redo Questions 1-4 in Part II — Subpart B based on your simulated values.