Stat 344 Homework Assignment 5 Solutions

1. You are using the Equivalence Principle to price a \$100,000 20 year term policy issued to (50). You are given the following:

$$A_{50:\overline{20|}} = 0.4 \qquad v = 0.95 \qquad {}_{20}p_{50} = 0.9$$

   (a) Assuming that the premiums are paid annually and the death benefit is paid at the end of the year of death, calculate the annual premium. [644.69]

   (b) Assuming that the premiums are paid monthly and the death benefit is paid at the moment of death, calculate the monthly premium. Use the UDD assumption where necessary. [56.58]

Answer:

(a) First calculate

$$A^{1}_{50:\overline{20|}} = A_{50:\overline{20|}} - A_{50:\overline{20|}}^{\ \ \ \ 1} = 0.4 - (0.95)^{20} \times 0.9 = 0.077$$

and

$$\ddot{a}_{50:\overline{20|}} = \frac{1 - A_{50:\overline{20|}}}{d} = \frac{0.6}{0.05} = 12$$

$$P = \frac{100,000 A^{1}_{50:\overline{20|}}}{\ddot{a}_{50:\overline{20|}}} = \frac{100,000(0.077)}{12} = \boxed{644.69}$$

(b)

$$P = \frac{100,000 \bar{A}^{1}_{50:\overline{20|}}}{\ddot{a}^{(12)}_{50:\overline{20|}}} \overset{UDD}{=} \frac{100,000(0.077)\frac{i}{\delta}}{12\alpha(12) - \beta(12)(1 - (0.95)^{20} \times 0.9)} = \boxed{56.58}$$

2. A person age (85) purchases a three year term policy, with gross premiums payable annually in advance and the death benefit payable at the end of the year of death. The death benefit is \$100,000 plus the sum of the gross premiums paid. You are given:

$$q_{85} = 0.2 \qquad q_{86} = 0.3 \qquad q_{87} = 0.4 \qquad i = 10\%$$

Expenses consist of:

- 30% of the initial gross premium and 2% of renewal gross premiums
- \$500 at issue

Using the equivalence principle, calculate the gross annual premium for this policy. [70703]

Answer:
EPV of the benefits are:

$$\frac{(100,000+P)(0.2)}{1.1}+\frac{(100,000+2P)(0.8)(0.3)}{(1.1)^2}+\frac{(100,000+3P)(0.8)(0.7)(0.4)}{(1.1)^3}$$

$= 54845.98 + 1.0834P$

EPV of the expenses are

$$500 + 0.28P + 0.02P(1 + 0.8/1.1 + (0.8)(0.7)/(1.1)^2) = 500 + 0.3238P$$

EPV of the premiums are

$$P(1 + 0.8/1.1 + (0.8)(0.7)/(1.1)^2) = 2.19P$$

Then the equation of value is

$$\begin{aligned} 54845.98 + 1.0834P + 500 + 0.3238P &= 2.19P \\ 0.7828P &= 55345.98 \\ P &= 70703 \end{aligned}$$

3. A 30-year old purchases a 3-year endowment insurance, with \$200,000 death benefit payable at the end of the year of death, and premium P payable at the beginning of each year while the contract is in force. You are given

$$p_{30} = 0.98, \qquad p_{31} = 0.97, \qquad p_{32} = 0.95, \qquad i = 10\%$$

(a) Write an expression for L_0^n.
Answer: $L_0^n = 200,000\, v^{\min(K_{30}+1,3)} - P\, \ddot{a}_{\overline{\min(K_{30}+1,3)}|}$

(b) Calculate the mean of L_0^n, in terms of P. Simplify as much as possible.
Answer: First we can calculate

$$\begin{aligned} A_{30:\overline{3}|} &= \left(v\, q_{30} + v^2\, p_{30}\, q_{31} + v^3\, p_{30}\, p_{31}\, q_{32}\right) + \left(v^3\, p_{30}\, p_{31}\, p_{32}\right) \\ &= \big((0.9091)(0.02) + (0.9091)^2(0.98)(0.03) + (0.9091)^3(0.98)(0.97)(0.05)\big) \\ &\qquad + \big((0.9091)^3(0.98)(0.97)(0.95)\big) \\ &= (0.07819) + (0.67849) \\ &= 0.75668 \end{aligned}$$

Then we have

$$
\begin{aligned}
E\left[L_0^n\right] &= E\left[200,000\, v^{\min(K_{30}+1,3)} - P\, \ddot{a}_{\overline{\min(K_{30}+1,3)|}}\right] \\
&= 200,000\, A_{30:\overline{3|}} - P\, \ddot{a}_{30:\overline{3|}} \\
&= 200,000\,(0.75668) - P\left(\frac{1-0.75668}{0.0909}\right) \\
&= 151,335.84 - 2.67653\, P
\end{aligned}
$$

(c) Calculate the variance of L_0^n, in terms of P. Simplify as much as possible.
Answer: First we can calculate

$$
\begin{aligned}
{}^2A_{30:\overline{3|}} &= \left(v^2\, q_{30} + (v^2)^2\, p_{30}\, q_{31} + (v^3)^2\, p_{30}\, p_{31}\, q_{32}\right) + \left((v^3)^2\, p_{30}\, p_{31}\, p_{32}\right) \\
&= \left((0.9091)^2(0.02) + (0.9091)^4(0.98)(0.03) + (0.9091)^6(0.98)(0.97)(0.05)\right) \\
&\qquad + \left((0.9091)^6(0.98)(0.97)(0.95)\right) \\
&= (0.06344) + (0.50976) \\
&= 0.57320
\end{aligned}
$$

Then we have

$$
\begin{aligned}
Var\left(L_0^n\right) &= Var\left(200,000\, v^{\min(K_{30}+1,3)} - P\, \ddot{a}_{\overline{\min(K_{30}+1,3)|}}\right) \\
&= Var\left(200,000\, v^{\min(K_{30}+1,3)} - P\left(\frac{1-v^{\min(K_{30}+1,3)}}{0.0909}\right)\right) \\
&= Var\left(200,000\, v^{\min(K_{30}+1,3)} - \frac{P}{0.0909} + \frac{P}{0.0909}\, v^{\min(K_{30}+1,3)}\right) \\
&= Var\left(\left(200,000 + \frac{P}{0.0909}\right) v^{\min(K_{30}+1,3)} - \frac{P}{0.0909}\right) \\
&= Var\left(\left(200,000 + \frac{P}{0.0909}\right) v^{\min(K_{30}+1,3)}\right) \\
&= \left(200,000 + \frac{P}{0.0909}\right)^2 Var\left(v^{\min(K_{30}+1,3)}\right) \\
&= \left(200,000 + \frac{P}{0.0909}\right)^2 \left({}^2A_{30:\overline{3|}} - (A_{30:\overline{3|}})^2\right) \\
&= \left(200,000 + \frac{P}{0.0909}\right)^2 (0.57320 - 0.57256) \\
&= \left(200,000 + \frac{P}{0.0909}\right)^2 (0.000637)
\end{aligned}
$$

(d) Calculate the annual premium P, assuming it has been determined using the equivalence principle. [56,541.83]

Answer:
We can set $E\left[L_0^n\right]$ from above equal to 0 and solve for P:
$0 = 151,335.84 - 2.67653\,P$ so that $P = 151,335.84/2.67653 = 56,541.83$.

4. Let L_0^n denote the present-value-of-loss random variable for a fully continuous whole life insurance policy issued to (x). Premiums are paid at a continuous rate of 0.09 per year and a benefit of amount 2 is paid at the moment of death. If $\delta = 0.06$ and $\mu_{x+t} = 0.04$ for all t, find $Var(L_0^n)$. [1.1025]

   Answer:

$$\begin{aligned} L_0^n &= 2v^{T_x} - 0.09\left(\frac{1-v^{T_x}}{0.06}\right) \\ &= 3.5v^{T_x} - 1.5 \end{aligned}$$

so that

$$\begin{aligned} Var(L_0^n) &= 3.5^2 Var(v^{T_x}) \\ &= (12.25)({}^2\overline{A}_x - \overline{A}_x^2) \\ &= (12.25)\left[\frac{0.04}{0.16} - \left(\frac{0.04}{0.1}\right)^2\right] \\ &= 1.1025 \end{aligned}$$

5. An insurer issues 100 fully discrete whole life policies to independent persons age (x). Assume that

$$d = 0.06 \qquad A_x = 0.4 \qquad {}^2A_x = 0.2$$

The policies are distributed as follows:

Face Amount	Number of Policies	Annual Premium Per Policy
100,000	80	5,000
400,000	20	19,000

Using a normal approximation, find the approximate probability that the present value of the insurer's profits exceeds 4,000,000. [0.00015]

Answer:
For each of the smaller policies,

$$\begin{aligned} L_0^s &= 100,000\,v^{K_x+1} - 5,000\left(\frac{1-v^{K_x+1}}{d}\right) \\ &= 183,333\,v^{K_x+1} - 83,333 \end{aligned}$$

Then

$$E[L_0^s] = 183,333\, A_x - 83,333 = -10,000$$

and

$$Var[L_0^s] = (183,333)^2 \left({}^2A_x - A_x^2\right) = 1,344,439,556.$$

Similarly, for each of the larger policies,

$$\begin{aligned} L_0^\ell &= 400,000\, v^{K_x+1} - 19,000 \left(\frac{1 - v^{K_x+1}}{d}\right) \\ &= 716,667\, v^{K_x+1} - 316,667 \end{aligned}$$

Then

$$E[L_0^\ell] = 716,667\, A_x - 316,667 = -30,000$$

and

$$Var[L_0^\ell] = (716,667)^2 \left({}^2A_x - A_x^2\right) = 20,544,463,556.$$

Thus, for the total block of policies,

$$E[L_0^{agg}] = 80(-10,000) + 20(-30,000) = -1,400,000$$

and

$$Var[L_0^{agg}] = 80(1,344,439,556) + 20(20,544,463,556) = 518,444,435,600$$

so that $SD(L_0^{agg}) = 720,031$. Then

$$P\left(L < -4000000\right) = P\left(Z > -\frac{2,600,000}{720,031}\right) = \boxed{0.00015}$$

6. An insurance company wants to use the Portfolio Percentile Premium Principle in order to set the annual premium amount P for whole life policies issued to x year olds. Assume that there are N insureds, all independent of one another. The death benefit for each policy will be \$500,000, payable at the end of the year of death. Assume that the insurer sets $\alpha = 0.95$ in its calculations. You are given that:

$$i = 6\% \qquad A_x = 0.3051431 \qquad {}^2A_x = 0.1306687$$

   (a) Find the premium P if $N = 100$. [14387.17]
   (b) Find the premium P if $N = 2500$. [12805.84]
   (c) Find the premium P if $N = 10000$. [12616.37]

(d) Find the premium under the Equivalence Principle. [12427.81]

Answer: We want to find the premium such that $P[L < 0]$ where L is the total future loss random variable. First we find the mean and variance of future loss for a single policy:

$$\begin{aligned} L_{0,i} &= 500,000\, v^{K_x+1} - P\, \ddot{a}_{\overline{K_x+1}|} \\ &= 500,000\, v^{K_x+1} - P \left(\frac{1 - v^{K_x+1}}{0.0566} \right) \\ &= 500,000\, v^{K_x+1} - \frac{P}{0.0566} + P\, \frac{v^{K_x+1}}{0.0566} \\ &= v^{K_x+1} \,(500,000 + 17.667\, P) - 17.667\, P \end{aligned}$$

$$\begin{aligned} E[L_{0,i}] &= E\left[v^{K_x+1} \,(500,000 + 17.667\, P) - 17.667\, P \right] \\ &= A_x \,(500,000 + 17.667\, P) - 17.667\, P \end{aligned}$$

$$\begin{aligned} Var[L_{0,i}] &= Var\left[v^{K_x+1} \,(500,000 + 17.667\, P) - 17.667\, P \right] \\ &= Var\left[v^{K_x+1} \,(500,000 + 17.667\, P) \right] \\ &= (500,000 + 17.667\, P)^2 \; Var\left[v^{K_x+1} \right] \\ &= (500,000 + 17.667\, P)^2 \; \left[{}^2A_x - (A_x)^2 \right] \end{aligned}$$

Then, for L, we have

$$E[L] = N \cdot (A_x \,(500,000 + 17.667\, P) - 17.667\, P)$$

and

$$Var(L) = N \cdot (500,000 + 17.667\, P)^2 \; \left[{}^2A_x - (A_x)^2 \right]$$

Substituting the values given yields:

$$E[L] = N \cdot ((0.3051431)\,(500,000 + 17.667\, P) - 17.667\, P)$$

and

$$Var(L) = N \cdot (500,000 + 17.667\, P)^2 \; \left[0.1306687 - (0.3051431)^2 \right]$$

We want to set $P[L < 0] = 0.95$. This implies that we should set $\dfrac{-E[L]}{\sqrt{Var(L)}} =$

1.645:

$$\frac{-E[L]}{\sqrt{Var(L)}} = 1.645$$

$$\frac{-N \cdot ((0.3051431)\,(500,000 + 17.667\,P) - 17.667\,P)}{\sqrt{N \cdot (500,000 + 17.667\,P)^2\,\left[0.1306687 - (0.3051431)^2\right]}} = 1.645$$

$$\frac{N \cdot (12.2758\,P - 152,571.55)}{\sqrt{N} \cdot (500,000 + 17.667\,P)\,\,[0.193795]} = 1.645$$

$$\frac{\sqrt{N} \cdot (12.2758\,P - 152,571.55)}{(96,897.353 + 3.423706\,P)} = 1.645$$

$$\sqrt{N} \cdot (12.2758\,P - 152,571.55) = 159,396.1466 + 5.631997\,P \tag{1}$$

Now, if we plug in $N = 100$ into (1), we can solve to get $\boxed{\text{P} = 14{,}387.17}$

If we plug in $N = 2500$ into (1), we can solve to get $\boxed{\text{P} = 12{,}805.84}$

If we plug in $N = 10000$ into (1), we can solve to get $\boxed{\text{P} = 12{,}616.37}$