Stat 344 — Fall 2023

Homework Assignment 5

Due Date: Thursday, December 14th in class

1. A person age x purchases a fully discrete 3-year term insurance. The death benefit for the policy is $100,000 plus the sum of all premiums paid. Use the following basis for all calculations:

$$q_x = 0.1, \qquad q_{x+1} = 0.2, \qquad q_{x+2} = 0.3, \qquad i = 10\%$$

(a) Calculate the net annual premium. [26135.78] **Answer:**

$$\begin{aligned} P\left(1 + \frac{0.9}{1.1} + \frac{(0.9)(0.8)}{(1.1)^2}\right) &= \frac{(100000+P)(0.1)}{1.1} \\ &\quad + \frac{(100000+2P)(0.9)(0.2)}{(1.1)^2} + \frac{(100000+3P)(0.9)(0.8)(0.3)}{(1.1)^3} \\ 1.537943P &= 40195.34 \\ P &= 26135.78 \end{aligned}$$

(b) Write a prospective formula for ${}_1V^n$ in terms of insurance and annuity symbols. Use this equation to calculate ${}_1V^n$. [17928.7] **Answer:**

$$\begin{aligned} {}_1V^n &= \frac{(100000+2P)(0.2)}{(1.1)} + \frac{(100000+3P)(0.8)(0.3)}{(1.1)^2} - 26135.78(1+0.8/1.1) \\ &= 17928.7 \end{aligned}$$

To check the answer, we can calculate it recursively:

$$\begin{aligned} (26135.78)(1.1) &= 0.1(126135.78) + 0.9\,{}_1V^n \\ {}_1V^n &= ((26135.78)(1.1) - 12613.58)/0.9 \\ {}_1V^n &= 17928.64. \end{aligned}$$

(c) Write an equation for ${}_2V^n$ in terms of ${}_1V^n$. Use this equation to solve for ${}_2V^n$. **Answer:**

$$\begin{aligned} ({}_1V^n + 26135.78)(1.1) &= (0.2)(100000 + 2(26135.78)) + (0.8)\,{}_2V^n \\ {}_2V^n &= ((17928.64 + 26135.78)(1.1) - (0.2)(100000 + 2(26135.78)))/(0.8) \\ {}_2V^n &= 22520.69 \end{aligned}$$

(d) Write an equation for ${}_2V^n$ in terms of ${}_3V^n$. Use this equation to solve for ${}_2V^n$.
Answer:

$$\begin{aligned}({}_2V^n + 26135.78)(1.1) &= (0.3)(100000 + 3(26135.78)) + (0.7)\,{}_3V^n \\ ({}_2V^n + 26135.78)(1.1) &= (0.3)(100000 + 3(26135.78)) + 0 \\ {}_2V^n &= ((0.3)(100000 + 3(26135.78)))/(1.1) - 26135.78 \\ {}_2V^n &= 22520.77\end{aligned}$$

2. A single premium continuous whole life annuity is issued to (65). The annuity pays at a rate of \$5,000 per year. The premium and policy value bases are:

- Mortality is modeled as a constant force of mortality, $\mu = 0.07$.
- Interest is given by $\delta = 0.03$.
- The only expenses are an amount of \$1,000 incurred at policy issue and amount of \$500 incurred at the policy termination.

(a) Calculate the gross single premium for this policy. [51,350]

(b) Calculate the gross premium policy value at time 5 for this policy. [50,350]

Answer: The gross premium is given by

$$G = 50000 + 1000 + 500\left(\frac{0.07}{0.1}\right) = 51,350.$$

$$\begin{aligned}{}_5V^g &= 5000\,\bar{a}_{70} + 500\,\bar{A}_{70} \\ &= 50,350\end{aligned}$$

3. A fully discrete whole life policy is issued to (45) whose death benefit is \$100,000 until the insured reaches age 65, and \$50,000 thereafter. Premiums are level and paid anually. Use the following basis for all calculations:

- Mortality follows the SULT.
- $i = 5\%$
- Expenses are:
  - 5% of all gross premiums
  - \$1,000 at the start of year 1; \$100 at the start of each subsequent year the policy is in force

(a) Calculate the net and gross premiums for this policy. [492.60, 676.96] **Answer:**

$$\begin{aligned}
P\,\ddot{a}_{45} &= 50000\,A_{45} + 50000\,A^{\;1}_{45:\overline{20}|}\\
P\,\ddot{a}_{45} &= 50000\,A_{45} + 50000\,(A_{45} - {}_{20}E_{45}\,A_{65})\\
17.8162\,P &= 50000(0.15161) + 50000\,(0.15161 - (0.35994)(0.35477))\\
P &= 492.60
\end{aligned}$$

$$\begin{aligned}
G\,\ddot{a}_{45} &= 50000\,A_{45} + 50000\,A^{\;1}_{45:\overline{20}|} + 0.05\,G\,\ddot{a}_{45} + 900 + 100\,\ddot{a}_{45}\\
0.95\,G\,\ddot{a}_{45} &= 50000\,A_{45} + 50000\,A^{\;1}_{45:\overline{20}|} + 900 + 100\,\ddot{a}_{45}\\
16.92539\,G &= 8776.20 + 900 + 1781.62\\
G &= 676.96
\end{aligned}$$

(b) Calculate ${}_{10}V^n$ and ${}_{10}V^g$ using prospective formulas. [5086.51, 4275.30] **Answer:**

$$\begin{aligned}
{}_{10}V^n &= 50000\,A_{55} + 50000\,A^{\;1}_{55:\overline{10}|} - P\,\ddot{a}_{55}\\
{}_{10}V^n &= 50000\,(0.23524) + 50000\,(0.23524 - 0.59342(0.35477)) - (492.60)\,(16.0599)\\
{}_{10}V^n &= 5086.51
\end{aligned}$$

$$\begin{aligned}
{}_{10}V^g &= 50000\,A_{55} + 50000\,A^{\;1}_{55:\overline{10}|} + 0.05\,G\,\ddot{a}_{55} + 100\,\ddot{a}_{55} - G\,\ddot{a}_{55}\\
{}_{10}V^g &= 12997.62 + 100\,\ddot{a}_{55} - 0.95\,G\,\ddot{a}_{55}\\
{}_{10}V^g &= 4275.30
\end{aligned}$$

(c) Calculate ${}_{11}V^n$ and ${}_{11}V^g$ using recursion formulas. [5670.07, 4869.74] **Answer:**

$$\begin{aligned}
({}_{10}V^n + P)(1+i) &= q_{55}(100000) + p_{55}\,{}_{11}V^n\\
(5086.51 + 492.60)(1.05) &= 199.3 + (1 - 0.001993)\,{}_{11}V^n\\
{}_{11}V^n &= 5670.07
\end{aligned}$$

$$\begin{aligned}
({}_{10}V^g + G - 0.05G - 100)(1+i) &= q_{55}(100000) + p_{55}\,{}_{11}V^g\\
(4275.30 + 0.95(676.96) - 100)(1.05) &= 199.3 + (1 - 0.001993)\,{}_{11}V^g\\
{}_{11}V^g &= 4869.74
\end{aligned}$$

4. A single premium whole life annuity-due is issued to (65) that pays a monthly benefit of $1,000. The first 10 years of payments are guaranteed. The only expenses are $700 at issue and $45 per year in renewal years. (Assume that the renewal expenses are incurred only while the policyholder is alive.) Assume that mortality is given by the SULT; use the UDD fractional age assumption where necessary. Also assume an annual effective interest rate of 5%.

(a) Calculate the net single premium for this policy. [160,546.62]

(b) Calculate the gross single premium for this policy. [161,811.40]

(c) Calculate ${}_8V^n$, ${}_8V^g$, and ${}_8V^e$ for this policy, assuming the insured is alive at that time. [126,850.76; 127,346.10; 495.36]

(d) Calculate ${}_{12}V^n$ for this policy. [109,798.95]

(e) Calculate ${}_{0.05}p_{76.95}$ for this person. [0.998946]

(f) Calculate ${}_{11.95}V^n$ for this policy. [109,415.96]

Answer:

(a)

$$\begin{aligned}
P &= 12,000\left[\ddot{a}^{(12)}_{\overline{10}|} + {}_{10}E_{65}\,\ddot{a}^{(12)}_{75}\right]\\
P &= 12,000\left[\ddot{a}^{(12)}_{\overline{10}|} + {}_{10}E_{65}\,\left(\alpha(12)\,\ddot{a}_{75} - \beta(12)\right)\right]\\
P &= 12,000\left[(7.929488) + (0.55305)\,\left((1.0002)\,(10.3178) - (0.46651)\right)\right]\\
P &= \boxed{160{,}546.62}
\end{aligned}$$

(b)

$$\begin{aligned}
G &= 12,000\left[\ddot{a}^{(12)}_{\overline{10}|} + {}_{10}E_{65}\,\ddot{a}^{(12)}_{75}\right] + 655 + 45\,\ddot{a}_{65}\\
G &= P + 655 + 45\,\ddot{a}_{65}\\
G &= 160,546.62 + 655 + 45\,(13.5498)\\
G &= \boxed{161{,}811.40}
\end{aligned}$$

(c) Prospectively compute the net and gross premium reserve, then the expense reserve is the difference:

$$\begin{aligned}
{}_8V^n &= 12,000\left(\ddot{a}^{(12)}_{\overline{2}|} + {}_2E_{73}\,\ddot{a}^{(12)}_{75}\right)\\
&= 12,000\left(1.909438 + \left(\frac{1}{(1.05)^2}\cdot\frac{85,203.5}{87,916.8}\right)\left((1.0002)\,(10.3178) - (0.46651)\right)\right)\\
&= \boxed{126{,}850.76}
\end{aligned}$$

$$\begin{aligned}
{}_8V^g &= {}_8V^n + 45\,\ddot{a}_{73}\\
&= 126,850.76 + 45(11.0081)\\
&= \boxed{127{,}346.10}
\end{aligned}$$

$${}_8V^e = \boxed{495.36}$$

(d)

$$
\begin{aligned}
{}_{12}V^n &= 12,000\, \ddot{a}_{77}^{(12)} \\
{}_{12}V^n &= 12,000\ (\alpha(12)\, \ddot{a}_{77} - \beta(12)) \\
{}_{12}V^n &= 12,000\ ((1.0002)\,(9.6145) - (0.46651)) \\
{}_{12}V^n &= \boxed{109{,}798.95}
\end{aligned}
$$

(e)

$$
\begin{aligned}
p_{76} &= {}_{0.95}p_{76}\ \cdot\ {}_{0.05}p_{76.95} \\
0.979332 &\overset{UDD}{=} (1 - (0.95) \cdot q_{76}) \cdot\ {}_{0.05}p_{76.95} \\
0.979332 &\overset{UDD}{=} (1 - (0.95)(0.020668)) \cdot\ {}_{0.05}p_{76.95} \\
{}_{0.05}p_{76.95} &\overset{UDD}{=} \boxed{0.998946}
\end{aligned}
$$

(f)

$$
\begin{aligned}
{}_{11.95}V^n &= v^{0.05}\ {}_{0.05}p_{76.95}\ {}_{12}V^n \\
{}_{11.95}V^n &= \frac{0.998946}{(1.05)^{0.05}} \cdot (109,798.95) \\
{}_{11.95}V^n &= \boxed{109{,}415.96}
\end{aligned}
$$

Answer:

(a) First we calculate

$$
\begin{aligned}
\overline{A}_{40} &= \int_0^{60} e^{-\delta t}\, \frac{1}{60}\, dt \\
\overline{A}_{40} &= \frac{1}{60} \int_0^{60} e^{-0.04879t}\, dt \\
\overline{A}_{40} &= \frac{1}{60}\, \left[-e^{-0.04879t}/0.04879\right]_0^{60} \\
\overline{A}_{40} &= 0.3233121
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{a}_{40} &= \int_0^{60} {}_tp_{40}\, e^{-\delta t}\, dt \\
\overline{a}_{40} &= \int_0^{60} (1 - t/60)\, e^{-\delta t}\, dt \\
\overline{a}_{40} &= \int_0^{60} e^{-\delta t} - t\, e^{-\delta t}/60\, dt \\
&= ((-13.4946 + 0.3416\, t))(e^{-0.04879\, t})\Big|_0^{60} \\
&= 13.86943
\end{aligned}
$$

so that

$$
\begin{aligned}
P\, \overline{a}_{40} &= 100,000\, \overline{A}_{40} \\
P &= \frac{100,000\, \overline{A}_{40}}{\overline{a}_{40}} \\
P &= \frac{100,000(0.3233121)}{13.86943} \\
P &= 2331.11
\end{aligned}
$$

(b) Prospectively,

$$
\begin{aligned}
{}_{15.7}V &= 100,000\, \overline{A}_{55.7} - P\, \overline{a}_{55.7} \\
{}_{15.7}V &= 40,938 - 2,331.11(12.10532) \\
{}_{15.7}V &= 12,719.17
\end{aligned}
$$

(c) For this mortality model

$$
S_0(x) = \frac{100 - x}{100} \qquad \text{and} \qquad -\, S_0'(x) = \frac{1}{100}
$$

so that

$$
\mu_x = \frac{1}{100 - x} \qquad \Rightarrow \qquad \mu_{55.7} = \frac{1}{100 - 55.7} = 0.022573
$$

(d)

$$
\begin{aligned}
\frac{d}{dt}\, {}_{15.7}V &= (0.04879)\, 12,719.17 + 2331.11 - (100,000 - 12,719.17)\, (0.022573) \\
\frac{d}{dt}\, {}_{15.7}V &= 981.488
\end{aligned}
$$

(e)

$$
\begin{aligned}
{}_{15.8}V &\approx {}_{15.7}V + 0.1\left((0.04879)12,719.17 + 2331.11 - (100,000 - 12,719.17)(0.022573)\right)\\
{}_{15.8}V &\approx 12,719.17 + 0.1\,(981.488)\\
{}_{15.8}V &\approx 12,817.32
\end{aligned}
$$

(f)

$$
\begin{aligned}
12,719.17 + P'\,\overline{a}_{55.7:\overline{34.3}|} &= 100,000\,\bar{A}_{55.7:\overline{34.3}|}\\
12,719.17 + P'\,\overline{a}_{55.7:\overline{34.3}|} &= 100,000\,(\bar{A}^{\;1}_{55.7:\overline{34.3}|} + {}_{34.3}E_{55.7})\\
12,719.17 + P'\;(11.92421) &= 100,000\left(0.3758727 + \frac{10}{(44.3)1.04^{34.3}}\right)\\
P' &= 2,440.64
\end{aligned}
$$

(g) (i) $12,719.17 - 200 = 12,519.17$

(ii)

$$12,719.17 = 100,000\,\bar{A}^{\;1}_{55.7:\overline{n}|} \qquad \Rightarrow \qquad n \approx 6.59 \text{ years}$$

(iii)

$$12,719.17 = B\,\overline{A}_{55.7} \qquad \Rightarrow \qquad B = 31069.35$$

5. Jill purchases a \$250,000 fully discrete 20-year term life insurance policy at age 35. Mortality is given by the SULT. Assume an annual effective interest rate of 5%. Expenses are as follows:

- Issue expenses of \$1,000.
- Maintenance expenses of \$50 per year in renewal years.
- Premium taxes of 2% of all gross premiums.
- Claims expenses of \$100 at the end of the year of death.

(a) Calculate ${}_5V^n$, ${}_5V^g$, and ${}_5V^e$ for this policy. [444.02, -347.06, -791.08]

(b) Calculate the FPT modified premiums for this policy. [187.63]

(c) Calculate ${}_5V^{FPT}$ for this policy. [366.02]

Answer:

(a) First we'll calculate the net and gross premiums:

$$P = \frac{250000\, A^{\,1}_{35:\overline{20|}}}{\ddot{a}_{35:\overline{20|}}}$$
$$P = \frac{250000\,(A_{35:\overline{20|}} - {}_{20}E_{35})}{\ddot{a}_{35:\overline{20|}}}$$
$$P = \frac{250000\,(0.37981 - 0.37041)}{13.024}$$
$$P = 180.44$$

$$G = \frac{250100\, A^{\,1}_{35:\overline{20|}} + 950 + 50\ddot{a}_{35:\overline{20|}}}{(0.98)\ddot{a}_{35:\overline{20|}}}$$
$$G = \frac{250100\,(0.37981 - 0.37041) + 950 + 50(13.024)}{0.98(13.024)}$$
$$G = 309.64$$

$$\begin{aligned}
{}_5V^n &= 250000\, A^{\,1}_{40:\overline{15|}} - P\,\ddot{a}_{40:\overline{15|}} \\
{}_5V^n &= 250000\,(A_{40} - {}_{10}E_{40}\,{}_5E_{50}\,A_{55}) - P\,(\ddot{a}_{40} - {}_{10}E_{40}\,{}_5E_{50}\,\ddot{a}_{55}) \\
{}_5V^n &= 250000\,(0.12106 - (0.6092)\,(0.77772)\,(0.23524)) \\
&\qquad - (180.44)\,(18.4578 - (0.6092)\,(0.77772)\,(16.0599)) \\
{}_5V^n &= 444.02
\end{aligned}$$

$$\begin{aligned}
{}_5V^g &= 250100\, A^{\,1}_{40:\overline{15|}} + 50\,\ddot{a}_{40:\overline{15|}} - (0.98)\,G\,\ddot{a}_{40:\overline{15|}} \\
{}_5V^g &= 250100\,(A_{40} - {}_{10}E_{40}\,{}_5E_{50}\,A_{55}) + (50 - 0.98\,G)\,(\ddot{a}_{40} - {}_{10}E_{40}\,{}_5E_{50}\,\ddot{a}_{55}) \\
{}_5V^g &= 250100\,(0.12106 - (0.6092)\,(0.77772)\,(0.23524)) \\
&\qquad + (50 - 0.98(309.64))\,(18.4578 - (0.6092)\,(0.77772)\,(16.0599)) \\
{}_5V^g &= -347.06
\end{aligned}$$

$${}_5V^e = {}_5V^g - {}_5V^n = -347.06 - 444.02 = -791.08$$

(b) First we calculate ${}_{19}E_{36} = \dfrac{97846.2}{(99517.8)\,(1.05)^{19}} = 0.389097.$

Then for this policy, $\alpha = 250000\, v\, q_{35} = 250000(0.000391)/1.05 = 93.10$ and

$$\begin{aligned}
\beta &= \frac{250000\, A^{1}_{36:\overline{19}|}}{\ddot{a}_{36:\overline{19}|}} \\
&= \frac{250000\,(A_{36} - {}_{19}E_{36}\, A_{55})}{\ddot{a}_{36} - {}_{19}E_{36}\, \ddot{a}_{55}} \\
&= \frac{250000\,(0.10101 - (0.389097)\,(0.23524))}{18.8788 - (0.389097)\,(16.0599)} \\
&= 187.63.
\end{aligned}$$

(c)

$$\begin{aligned}
{}_5V^{FPT} &= 250000\, A^{1}_{40:\overline{15}|} - \beta\, \ddot{a}_{40:\overline{15}|} \\
&= 250000(A_{40} - {}_{10}E_{40}\, {}_5E_{50}\, A_{55}) - (187.63)(\ddot{a}_{40} - {}_{10}E_{40}\, {}_5E_{50}\, \ddot{a}_{55}) \\
&= 250000\,(0.12106 - (0.6092)\,(0.77772)\,(0.23524)) \\
&\qquad - (187.63)\,(18.4578 - (0.6092)\,(0.77772)\,(16.0599)) \\
&= 366.02
\end{aligned}$$

6. Time to death from the onset of a disease in a study of 10 lives finds the following sample:

   1, 2, 4, 4, 5, 6, 6, 7, 10, 15

   (a) Find $S(5.5)$
      i. Using Kaplan-Meier [.5]
      ii. Using Nelson-Åalen [.534]

   Now assume that 6 other people left the study for causes other than death at times:

   3, 3, 4, 6, 8, 10.

   (b) Find $S(5.5)$
      i. Using Kaplan-Meier [.6482]
      ii. Using Nelson-Åalen [.6657]

   (c) Calculate the variance of the two estimates in (b). [KM = 0.0166, NA = 0.0135]

   (d) Calculate the 95% linear confidence interval of the NA survival estimate in (b). [(0.5497, 0.7817)]

   (e) Calculate the 95% log-confidence interval for the KM survival estimate in (b). [(0.3449, 0.8381)]