Stat 344
Life Contingencies I

Chapter 2: Survival models

The future lifetime random variable — Notation

- We are interested in analyzing and describing the future lifetime of an individual.
- We use (x) to denote a life age x for $x \geq 0$.
- Then the *random variable* describing the future lifetime, or time until death, for (x) is denoted by T_x for $T_x \geq 0$.
- We also have notation for the density, distribution and survival functions of T_x:

Density: $f_x(t) = \frac{d}{dt} F_x(t)$

Distribution: $F_x(t) = Pr[T_x \leq t]$

Survival: $S_x(t) = Pr[T_x > t] = 1 - F_x(t)$

The future lifetime random variable — Relationships

We now consider the set of random variables $\{T_x\}_{x \geq 0}$ and the relationships between them:

Important Assumption

$$Pr[T_x \leq t] = Pr[T_0 \leq x + t | T_0 > x]$$

Important Consequence

$$S_0(x + t) = S_0(x) \cdot S_x(t)$$

This idea can be generalized:

Important General Relationship

$$S_x(t + u) = S_x(t) \cdot S_{x+t}(u)$$

Survival function conditions and assumptions

We will require any survival function corresponding to a future lifetime random variable to satisfy the following conditions:

Necessary and Sufficient Conditions

1. $S_x(0) = 1$
2. $\lim_{t \to \infty} S_x(t) = 0$
3. $S_x(t)$ must be a non-increasing function of t

In addition, we will make the following assumptions for the survival functions used in this course:

Assumptions

1. $S_x(t)$ is differentiable for all $t > 0$.
2. $\lim_{t \to \infty} t \cdot S_x(t) = 0$
3. $\lim_{t \to \infty} t^2 \cdot S_x(t) = 0$

Future lifetime random variable example

Assume that the survival function for a newborn is given by

$$S_0(t) = 8(t+2)^{-3}$$

1. Verify that this is a valid survival function.
2. Find the density function associated with the future lifetime random variable T_0, that is, find $f_0(t)$.
3. Find the probability that a newborn dies between the ages of 1 and 2.
4. Find the survival function corresponding to the random variable T_{10}.

Force of Mortality

- We will define the **force of mortality** as

$$\mu_x = \lim_{dx \to 0^+} \frac{Pr\left[T_0 \leq x + dx \middle| T_0 > x\right]}{dx}$$

- Then for small values of dx, we can use the approximation

$$\mu_x dx \approx Pr\left[T_0 \leq x + dx \middle| T_0 > x\right]$$

- We can relate the force of mortality to the survival function:

$$\mu_x = \frac{-\frac{d}{dx} S_0(x)}{S_0(x)} \quad \text{and} \quad S_x(t) = \exp\left\{-\int_0^t \mu_{x+s}\, ds\right\}$$

- An alternative relationship that's sometimes more convenient:

$$\mu_x = -\frac{d}{dx} \ln S_0(x)$$

Some commonly used forms for the force of mortality

It is sometimes convenient to describe the future lifetime random variable by explicitly modeling the force of mortality. A couple of the parametric forms that have been used are:

- Gompertz: $\mu_x = Bc^x, \;\; 0 < B < 1, \, c > 0$
- Makeham: $\mu_x = A + Bc^x, \;\; 0 < B < 1, \, c > 0$

If there is a **limiting age**, it is typically denoted by ω.

Force of mortality example

Again, assume that the survival function for a newborn is given by

$$S_0(x) = 8(x+2)^{-3}$$

1. Find the force of mortality, μ_x.
2. Consider the shape of μ_x as a function of x.
   - Does it seem to be a realistic survival model for describing human lifetimes?
   - What would you expect the force of mortality corresponding to humans to look like?

Actuarial Notation

Thus far we have described the properties of the future lifetime random variable using statistical notation. Actuaries have also developed their own (different) notation to describe many of the same concepts:

- ${}_tp_x = Pr\left[T_x > t\right] = S_x(t)$
- ${}_tq_x = Pr\left[T_x \leq t\right] = F_x(t)$
- ${}_{u|t}q_x = Pr\left[u < T_x \leq u + t\right] = F_x(u + t) - F_x(u)$

In any of the above expressions, the subscript t may be omitted when its value is 1.

The density of the random variable T_x can be written as $f_X(t) = {}_tp_x\,\mu_{x+t}$.

Some Basic Relationships

We can express some relationships in this actuarial notation:

- ${}_tp_x = 1 - {}_tq_x$
- ${}_{u|t}q_x = {}_up_x - {}_{u+t}p_x$
- ${}_{t+u}p_x = {}_tp_x \, {}_up_{x+t}$
- ${}_{u|t}q_x = {}_up_x \, {}_tq_{x+u}$
- ${}_tp_x = \exp\left\{-\int_0^t \mu_{x+s}\, ds\right\}$
- ${}_tq_x = \int_0^t {}_sp_x \, \mu_{x+s}\, ds$

Examples

1. ${}_5p_{30} = 0.9, \quad {}_5p_{35} = 0.8. \quad$ Find ${}_{10}p_{30}$.
2. ${}_3p_{80} = 0.6, \quad {}_2p_{81} = 0.7. \quad$ Find p_{80}.
3. ${}_{2|}q_{60} = 0.216, \quad {}_2p_{61} = 0.56, \quad p_{62} = 0.7. \quad$ Find q_{60}.
4. Show that $\frac{d}{dt}\, {}_tp_x = -{}_tp_x\, \mu_{x+t}$.

Mean of the Future Lifetime Random Variable

We can consider various properties of the future lifetime RV:

- One quantity that is often of interest is the mean of the future lifetime random variable, $E[T_x]$.
- The actuarial symbol for this mean is $\overset{\circ}{e}_x$:

$$\overset{\circ}{e}_x = \int_0^\infty t\, f_x(t)\, dt = \int_0^\infty t\, {}_tp_x\, \mu_{x+t}\, dt = \int_0^\infty {}_tp_x\, dt$$

- A related quantity is $\overset{\circ}{e}_{x:\overline{n}|} = \int_0^n {}_tp_x\, dt$

Variance of the Future Lifetime Random Variable

In order to calculate the variance of T_x, we need its second moment:

$$\begin{aligned} E\left[T_x^2\right] &= \int_0^\infty t^2\, f_x(t)\, dt \\ &= \int_0^\infty t^2\, {}_tp_x\, \mu_{x+t}\, dt \\ &= 2\int_0^\infty t\, {}_tp_x\, dt \end{aligned}$$

Then we can calculate the variance in the usual way:

$$\begin{aligned} V\left[T_x\right] &= E\left[T_x^2\right] - \left(E\left[T_x\right]\right)^2 \\ &= E\left[T_x^2\right] - \left(\overset{\circ}{e}_x\right)^2 \end{aligned}$$

Example

You are given the following survival function for a newborn:

$$S_0(t) = \frac{1}{10}\sqrt{100 - t}, \qquad 0 \leq t \leq 100$$

1. Find $\mathring{e}_{19}$.
2. Calculate $Var(T_{19})$.
3. Find the mode of T_{19}.
4. Verify that $S_0(t)$ is a valid survival function.

Summary of notation for the future lifetime RV

We can summarize the statistical and actuarial notation for the properties of T_x covered thus far:

Concept	Statistical Notation	Actuarial Notation
Expected Value	$E\left[T_x\right]$	$\overset{\circ}{e}_x$
Distribution Function	$F_x(t)$	${}_tq_x$
Survival Function	$S_x(t)$	${}_tp_x$
Force of Mortality	$\lambda(x)$ or λ_x	μ_x
Density	$f_x(t)$	${}_tp_x\,\mu_{x+t}$
Deferred Mortality Prob.	$Pr\left[u < T_x \leq u+t\right]$	${}_{u\vert t}q_x$

Curtate Future Lifetime Random Variable

We are often interested in the *integral* number of years lived in the future by an individual. This *discrete* random variable is called the **curtate future lifetime** and is denoted by K_x.

We can consider the pmf (probability mass function) of K_x:

$$\begin{aligned} Pr\left[K_x = k\right] &= Pr\left[k \leq T_x < k+1\right] \\ &= {}_{k|}q_x \\ &= {}_k p_x \, q_{x+k} \end{aligned}$$

We also have the relation $K_x = \lfloor T_x \rfloor$.

Moments of the Curtate Future Lifetime Random Variable

We can find the mean (denoted by e_x) and variance of K_x:

$$E[K_x] = e_x = \sum_{k=1}^{\infty} {}_k p_x$$

$$E\left[K_x^2\right] = 2\sum_{k=1}^{\infty} k\ {}_k p_x - e_x$$

$$V[K_x] = E\left[K_x^2\right] - (e_x)^2$$

There's a term version of this as well: $e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_k p_x$

Using the trapezoidal rule, we can also find an approximate relation between the means of T_x and K_x:

$$\mathring{e}_x \approx e_x + \frac{1}{2}$$

Example

You are given the following mortality information:

$$q_{90} = 0.1, \quad q_{91} = 0.2, \quad q_{92} = 0.3, \quad q_{93} = 1$$

1. Calculate $Pr(K_{90} = 1)$.
2. Calculate e_{90}.
3. Calculate $Var(K_{90})$.
4. Calculate $e_{90:\overline{2}|}$.
5. Write the pmf of K_{90}.
6. Repeat $1 - 5$, using the same mortality information, but substituting 91 for 90.

m^{th}ly future lifetime random variable

We define the *discrete* random variable $K_x^{(m)}$ to be the future lifetime, rounded down to the lower $1/m$ of a year. Some examples:

T_x	K_x	$K_x^{(2)}$	$K_x^{(4)}$	$K_x^{(12)}$
25.7	25	$25\frac{1}{2}$	$25\frac{2}{4}$	$25\frac{8}{12}$
25.1	25	25	25	$25\frac{1}{12}$

Note: In this definition, m must be an integer greater than 1.

The pmf of $K_x^{(m)}$ is similar to that of K_x:

$$Pr\left[K_x^{(m)} = k\right] = Pr\left[k \leq T_x < k + \frac{1}{m}\right] = {}_{k|\frac{1}{m}}q_x \quad k = 0, \frac{1}{m}, \frac{2}{m}, \ldots$$