Stat 344
Life Contingencies I

Chapter 5: Life annuities

Life annuities

Here we're going to consider the valuation of **life annuities**.

- A life annuity is regularly (e.g., continuously, annually, monthly, etc.) spaced series of payments, which are usually based on the survival of the policyholder.
- These annuities are important for retirement plans, pensions, structured settlements, life insurance, and in many other contexts.

Many of the ideas and notation used in the valuation of life annuities borrow from annuities-certain.

- As with life insurance, the life-contingent nature of the payments means that the present value of benefits is a *random variable* rather than a fixed number.

Review of annuities-certain

Recall the actuarial symbols for the present value of n-year annuities-certain:

Annuity-immediate: $a_{\overline{n|}} = \sum_{k=1}^{n} v^k = \dfrac{1 - v^n}{i}$

Annuity-due: $\ddot{a}_{\overline{n|}} = \sum_{k=0}^{n-1} v^k = \dfrac{1 - v^n}{d}$

Continuous annuity: $\bar{a}_{\overline{n|}} = \int_0^n e^{-\delta t}\, dt = \dfrac{1 - v^n}{\delta}$

m^{th}ly annuity-immediate: $a_{\overline{n|}}^{(m)} = \sum_{k=1}^{nm} \dfrac{1}{m} v^{k/m} = \dfrac{1 - v^n}{i^{(m)}}$

Review of annuities-certain — continued

Increasing annuity-immediate:

$$(Ia)_{\overline{n}|} = \sum_{k=1}^{n} k\, v^k = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

Decreasing annuity-immediate:

$$(Da)_{\overline{n}|} = \sum_{k=1}^{n} (n+1-k)\, v^k = \frac{n - a_{\overline{n}|}}{i}$$

For the corresponding accumulated values, we replace a by s in each case. For example:

$$a_{\overline{n}|}\,(1+i)^n = s_{\overline{n}|}$$

Whole life annuity-due

Consider an annuity that pays (x) an amount of \$1 on an annual basis for as long as (x) is alive, with the first payment occuring immediately.

- This type of life annuity is known as a **whole life annuity-due**.

We'll denote the random variable representing the PV of this annuity benefit by Y:

$$Y = 1 + v + v^2 + \cdots + v^{K_x} = \ddot{a}_{\overline{K_x+1}|} \qquad (1)$$

The pmf of Y is given by:

$$P\left(Y = \ddot{a}_{\overline{k}|}\right) = {}_{k-1}|q_x \quad \text{for } k = 1, 2, 3, \ldots$$

Whole life annuity-due EPV

The EPV of this annuity is denoted by $\ddot{a}_x$ and we can use our general strategy to find this EPV:

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \, {}_k p_x$$

We could also take the expectation of the expression given in (1):

Important Relation

$$\ddot{a}_x = E[Y] = \frac{1 - A_x}{d} \qquad \text{so that} \qquad 1 = d\,\ddot{a}_x + A_x$$

This relationship actually holds between the underlying random variables, not just the expected values.

Whole life annuity-due EPV, variance, and recursion

Finally, we could calculate the EPV using the standard formula for the expectation of a discrete RV:

$$\ddot{a}_x = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} \, {}_{k|}q_x$$

We can find the variance of Y in a similar manner:

$$V[Y] = \frac{{}^2A_x - (A_x)^2}{d^2}$$

(Note that we **cannot** find the second moment of Y by computing the first moment at double the force of interest.)

Further, we have the recursion $\ddot{a}_x = 1 + v\, p_x\, \ddot{a}_{x+1}$

Term annuity-due

Now consider an annuity that pays (x) an amount of \$1 on an annual basis for up to n years, so long as (x) is alive, with the first payment occuring immediately.

- This type of life annuity is known as a **term annuity-due**.

For this type of annuity, PV of the benefit is:

$$Y = \ddot{a}_{\overline{\min(K_x+1,\, n)}|} = \begin{cases} \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x \leq n-1 \\ \ddot{a}_{\overline{n}|} & \text{if } K_x \geq n \end{cases}$$

EPV for Term Annuity-Due

$$\ddot{a}_{x:\overline{n}|} = \frac{1 - A_{x:\overline{n}|}}{d} = \sum_{t=0}^{n-1} v^t \, {}_t p_x = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} \, {}_{k|}q_x + {}_n p_x \, \ddot{a}_{\overline{n}|}$$

$$Var(Y) = \frac{{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2}{d^2}$$

Example

Suppose that a person currently age 65 wants to purchase a 3-year term annuity due with annual payments of \$50,000 each. You are given that

$$p_{65} = 0.95 \qquad p_{66} = 0.91 \qquad p_{67} = 0.87 \qquad i = 7\%$$

Letting Y denote the PV of the annuity benefit, calculate:

(a) $E[Y]$
(b) $\mathrm{sd}(Y)$
(c) $Pr(Y > 70000)$

Now suppose that a life insurance company sells this type of annuity to 100 such (independent) people, all age 65. Letting S denote the total PV of the annuity benefits, calculate:

(a) $E[S]$
(b) $\mathrm{sd}(S)$
(c) $Pr(S > 1,300,000)$

Whole life annuity-immediate

Now we'll turn to consider life annuities-immediate, starting with a whole life annuity-immediate.

- This type of annuity pays (x) an amount of \$1 on an annual basis for as long as (x) is alive, with the first payment occuring at age $x + 1$.

We'll denote the random variable representing the PV of this annuity benefit by Y^*:

$$Y^* = v + v^2 + \cdots + v^{K_x} = a_{\overline{K_x}|}$$

with pmf

$$P\left(Y^* = a_{\overline{k}|}\right) = {}_{k}|q_x \quad \text{for } k = 0, 1, 2, 3, \ldots$$

Whole life annuity-immediate (continued)

We have the relationship $Y^* = Y - 1$, where Y is the PV of the whole life annuity-due and Y^* is the corresponding PV of the corresponding whole life annuity-immediate.

- The only difference between these two annuities is the payment at time 0.
- From this relationship, we can obtain expressions for the EPV and variance of Y^*

$$a_x = \ddot{a}_x - 1 \qquad\qquad V[Y^*] = V[Y] = \frac{{}^2A_x - (A_x)^2}{d^2}$$

We also have the recursion relation: $a_x = v\, p_x + v\, p_x\, a_{x+1}$

Term annuity-immediate

Now consider an annuity that pays (x) an amount of \$1 on an annual basis for up to n years, so long as (x) is alive, with the first payment occuring at age $x+1$.

- This type of life annuity is known as a **term annuity-immediate**.

For this type of annuity, PV of the benefit is:

$$Y^* = a_{\overline{\min(K_x, n)}|} = \begin{cases} a_{\overline{K_x}|} & \text{if } K_x \leq n \\ a_{\overline{n}|} & \text{if } K_x > n \end{cases}$$

In this case, the EPV is denoted by $a_{x:\overline{n}|}$

EPV for Term Annuity-Due

$$a_{x:\overline{n}|} = \sum_{t=1}^{n} v^t \, {}_tp_x = \ddot{a}_{x:\overline{n}|} + {}_nE_x - 1$$

Example

Assume that ${}_{20}E_x = 0.35, {}_{20}q_x = 0.3,$ and $A^{1}_{x:\overline{20|}} = 0.2.$

Find:

1. $A_{x:\overline{20|}}$
2. $\ddot{a}_{x:\overline{20|}}$
3. $a_{x:\overline{20|}}$

Whole life continuous annuity

Next we'll consider continuous life annuities, starting with a whole life continuous annuity.

- This type of annuity pays (x) continuously at a rate of \$1 per year for as long as (x) is alive, starting now.

Letting Y denote the random variable representing the PV of this annuity benefit, we have: $Y = \bar{a}_{\overline{T_x}|}$

EPV for Whole Life Continuous Annuity

$$\bar{a}_x = \int_0^\infty e^{-\delta t}\, {}_tp_x \, dt = \frac{1 - \bar{A}_x}{\delta} = \int_0^\infty \bar{a}_{\overline{t}|}\, {}_tp_x\, \mu_{x+t}\, dt$$

$$Var(Y) = \frac{{}^2\bar{A}_x - \left(\bar{A}_x\right)^2}{\delta^2}$$

Term continuous annuity

We can also have a continuous annuity that pays for a maximum of n years, so long as (x) is alive.

- This type of life annuity is known as an ***n*-year term continuous annuity**.

For this type of annuity, PV of the benefit is:

$$Y = \bar{a}_{\overline{\min(T_x,\, n)}|} = \begin{cases} \bar{a}_{\overline{T_x}|} & \text{if } T_x \leq n \\ \bar{a}_{\overline{n}|} & \text{if } T_x > n \end{cases}$$

EPV for *n*-year Term Continuous Annuity

$$\bar{a}_{x:\overline{n}|} = \int_0^n e^{-\delta t}\, {}_tp_x \, dt = \frac{1 - \bar{A}_{x:\overline{n}|}}{\delta} = \int_0^n \bar{a}_{\overline{t}|}\, {}_tp_x\, \mu_{x+t}\, dt + \bar{a}_{\overline{n}|}\, {}_np_x$$

$$Var(Y) = \frac{{}^2\bar{A}_{x:\overline{n}|} - \left(\bar{A}_{x:\overline{n}|}\right)^2}{\delta^2}$$

Example

Suppose that the survival model for (x) is described by the exponential model with $\mu_x = \mu \; \forall x$.

Let Y be the PV of a continuous whole life annuity payable at a rate of 1 per year to (x). Assume that $\delta = 0.08$ and $\mu = 0.04$.

1. Find the expected value and variance of Y.
2. Find the probability that $Y < 5$.
3. Consider $\bar{a}_x$ as a function of μ and as a function of δ. How does $\bar{a}_x$ change as each of these parameters increases (holding the other one constant)?

m^{th}ly life annuities

We can also analyze life annuities that pay on an m^{th}ly basis.

First consider an annuity that pays an amount of 1 per year, paid in installments of amount $\frac{1}{m}$ at the beginning of each m^{th} of a year, so long as (x) lives.

The PV of this annuity is $Y = \ddot{a}^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}}|} = \dfrac{1 - v^{K_x^{(m)}+\frac{1}{m}}}{d^{(m)}}$

The EPV of Y is given by

$$E[Y] = \ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}} = \sum_{k=0}^{\infty} \frac{1}{m}\, v^{\frac{k}{m}}\ {}_{\frac{k}{m}}p_x$$

We can also consider the "immediate" version: $a_x^{(m)} = \ddot{a}_x^{(m)} - \dfrac{1}{m}$

Now consider an annuity that pays an amount of 1 per year, paid in installments of amount $\frac{1}{m}$ at the beginning of each m^{th} of a year, so long as (x) lives, but for a maximum of n years.

The PV of this annuity is

$$Y = \ddot{a}^{(m)}_{\overline{\min\left(K_x^{(m)}+\frac{1}{m},\, n\right)}|} = \frac{1 - v^{\min\left(K_x^{(m)}+\frac{1}{m},\, n\right)}}{d^{(m)}}$$

The EPV of Y is given by

$$E[Y] = \ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1 - A^{(m)}_{x:\overline{n}|}}{d^{(m)}} = \sum_{k=0}^{mn-1} \frac{1}{m}\, v^{\frac{k}{m}}\ {}_{\frac{k}{m}}p_x$$

We can also consider the "immediate" version:

$$a^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x:\overline{n}|} + {}_nE_x\,\frac{1}{m} - \frac{1}{m}$$

Deferred annuities

As we did with life insurance, we can consider the notion of deferring annuity benefits.

- In this case, the first annuity payment would occur at some point in the future (rather than at the beginning or end of the first year).

We denote this deferment in the "usual" manner, so that, for example, the EPV of a whole life annuity due for (x), deferred u years, would be denoted by

$$ {}_{u|}\ddot{a}_x = \sum_{k=u}^{\infty} v^k \; {}_k p_x $$

The same idea also applies to the continuous, immediate, and m^{th}ly life annuity forms.

Deferred annuities (continued)

As we did with life insurance, we can use the idea of deferment to construct or deconstruct various types of annuities.

This leads to various relationships among the annuities and their EPVs:

$$ {}_{u|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{u}|} \qquad {}_{u|}\ddot{a}_x = {}_uE_x\, \ddot{a}_{x+u} $$

$$ \ddot{a}_{x:\overline{n}|} = \ddot{a}_x - {}_nE_x\, \ddot{a}_{x+n} \qquad \ddot{a}_{x:\overline{n}|} = \sum_{u=0}^{n-1} {}_{u|}\ddot{a}_{x:\overline{1}|} $$

Again, there are analogous relationships for the other annuity payment forms.

Example

You are given that:

$$\ddot{a}_{30:\overline{10}|} = 7.79064 \qquad \ddot{a}_{40:\overline{10}|} = 7.78144 \qquad \ddot{a}_{50} = 15.1511$$

$$_{10}p_{30} = 0.99611 \qquad _{10}p_{40} = 0.99233 \qquad i = 0.06$$

Find:

1. $_{10}E_{30}$
2. $\ddot{a}_{30:\overline{20}|}$
3. $\ddot{a}_{30}$

Guaranteed annuities

We can also consider annuities in which some of the payments are certain, rather than being contingent on the policyholder being alive at the time of payment.

- That is, some of the payments may be **guaranteed** rather than life contingent.
- This creates a sort of annuity that's a hybrid between the ones we've studied in this chapter and annuities-certain.
- Any payments made after the annuitant dies would go to a beneficiary.
- This reduces the risk of a very poor return, but there's a cost associated with it...

For example, consider a whole life annuity due that will pay for a minimum of n years, even if death occurs within the first n years.

- This is sometimes called a "life and n-year certain" annuity.

Guaranteed annuities (continued)

The PV of this annuity benefit would be

$$Y = \begin{cases} \ddot{a}_{\overline{n}|} & \text{if } K_x \le n-1 \\ \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x \ge n \end{cases}$$

The EPV of an whole life annuity due that guarantees the first n payments is denoted by $\ddot{a}_{\overline{x:\overline{n}|}}$

We can relate this EPV to other annuity EPVs:

$$\ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} + {}_nE_x\, \ddot{a}_{x+n}$$

Example

A person age 65 has accumulated a sum of $\$100,000$ in her retirement account. She wishes to use this money to purchase an level payment annuity, with the first payment occurring today.

Assume that mortality is given by the SULT and $i = 5\%$.

1. What payment amount could she afford if she purchases a whole life annuity due?

2. What payment amount could she afford if she purchases a whole life annuity due with the first 10 years guaranteed?

Annuities with varying benefits

We can also consider annuities with non-level payment patterns.

For example, we can use our general EPV strategy to find the EPV of an arithmetically increasing n-year term life annuity-due:

$$(I\ddot{a})_{x:\overline{n}|} = \sum_{t=0}^{n-1} v^t \, (t+1) \, {}_t p_x$$

The continuous version would be derived analogously:

$$\left(\bar{I}\bar{a}\right)_{x:\overline{n}|} = \int_0^n t \, e^{-\delta t} \, {}_t p_x \, dt$$

Fractional age assumptions

By using the fractional age relationships we developed for life insurances, we can find corresponding formulas for annuities. For example, under UDD we have:

$$\ddot{a}_x^{(m)} \overset{UDD}{=} \alpha(m)\,\ddot{a}_x - \beta(m)$$

where

$$\alpha(m) = \frac{id}{i^{(m)}\,d^{(m)}} \qquad \text{and} \qquad \beta(m) = \frac{i - i^{(m)}}{i^{(m)}\,d^{(m)}}$$

Letting $m \to \infty$ yields: $$\bar{a}_x \overset{UDD}{=} \left(\frac{id}{\delta^2}\right)\ddot{a}_x - \left(\frac{i-\delta}{\delta^2}\right)$$

And for an n-year term annuity, we have

$$\ddot{a}_{x:\overline{n}|}^{(m)} \overset{UDD}{=} \alpha(m)\,\ddot{a}_{x:\overline{n}|} - \beta(m)\,(1 - {}_nE_x)$$

Woolhouse's Formula

Another approach to approximating EPVs of $m^{th}ly$ and continuous life annuities is by using **Woolhouse's formula**. For an $m^{th}ly$ whole life annuity, the EPV approximation is given by

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)$$

Letting $m \to \infty$ yields: $\bar{a}_x \approx \ddot{a}_x - \frac{1}{2} - \frac{1}{12}(\delta + \mu_x)$

And for an n-year term life annuity, we have

$$\ddot{a}_{x:\overline{n}|}^{(m)} \approx \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - v^n {}_np_x) - \frac{m^2-1}{12m^2}(\delta + \mu_x - v^n {}_np_x(\delta + \mu_{x+n}))$$

and

$$\bar{a}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|} - \frac{1}{2}(1 - v^n {}_np_x) - \frac{1}{12}(\delta + \mu_x - v^n {}_np_x(\delta + \mu_{x+n}))$$

Examples

1. Consider a 20-year term annuity due issued to (x) with a first annual payment of $\$1,000$, increasing by 2% per year. Find an expression for the EPV of this annuity.

2. Calculate $\ddot{a}^{(12)}_{65:\overline{25}|}$ assuming SULT and $i = 5\%$, using the 3-term Woolhouse formula.

3. You are given:

$$q_{69} = 0.02 \qquad i = 0.05 \qquad \bar{A}_{70} = 0.53$$

Calculate $\ddot{a}^{(2)}_{69}$ under UDD.