Stat 444 — Winter 2024

Homework Assignment 1

Due Date: Thursday, January 25th at 2:00 pm

General Notes:

- For Part I, you may submit your assignment to Learning Suite or in class.
- For Part II, you should use a spreadsheet.
  - Submit electronically the Excel spreadsheet you create to answer the questions in Part II to Learning Suite. Your spreadsheet should be neatly organized and labeled; each answer should be highlighted in some manner, and it should be very clear how each of your answers was obtained.

Part I

1. Consider a 4 state discrete time model with states *Healthy*, *Temporarily Disabled*, *Permanently Disabled*, and *Dead*. The annual probability transition matrix given by:

$$\begin{array}{c c} & \begin{array}{cccc} H & T & P & D \end{array} \\ \begin{array}{c} H \\ T \\ P \\ D \end{array} & \begin{pmatrix} 0.85 & 0.09 & 0.05 & 0.01 \\ 0.25 & 0.6 & 0.1 & 0.05 \\ 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

For the following probability calculations, assume that we're currently at the beginning of a year and that transitions happen at the end of each year. Show all work and carry calculations to at least 5 decimal places.

(a) Calculate the probability that a person who is permanently disabled will be dead two years from now. [0.36]

(b) Calculate the probability that a person who is currently temporarily disabled will be disabled (temporarily or permanently) two years from now. [0.535]

(c) Calculate the probability that a person who is currently healthy will be healthy three years from now. [0.665875]

(d) Calculate the probability that a person who is currently healthy will be dead three years from now, having been temporarily disabled at some point in the interim. [0.01305]

(e) Consider a two year term insurance policy with \$100,000 death benefit issued to Kim, who is currently healthy. The benefit is paid upon transition to the dead state at the end of years 1 or 2. Use $i = 10\%$. Expenses consist of 5% of gross premiums.

  (i) Calculate the EPV of this life insurance. (Just consider the death benefit for this part; ignore expenses.) [2809.92]

  (ii) Find the gross level annual premium for this insurance, payable at the start of each year that Kim is alive. [1556.74]

  (iii) Calculate ${}_1V^g$ for this policy, for each state. [-569.81, 3066.55, 16702.92, 0]

  (iv) Now suppose that a disability benefit is added to the policy; in addition to the death benefit, the policy would now provide a one-time payment of \$10,000 upon Kim becoming permanently disabled. Also, Kim would not be required to pay a premium at time 1 if she is disabled (temporarily or permanently) at that time. Repeat parts (i), (ii), and (iii) for this new policy. [3690.08, 2191.14, -717.95, 5454.55, 18181.82, 0]

Answer:

(a) $P \to P \to D : (0.8)(0.2) = 0.16$
$P \to D \to D : (0.2)(1) = 0.2$
Total Prob. $= \boxed{0.36}$

(b) $T \to T \to T : (0.6)(0.6) = 0.36$
$T \to T \to P : (0.6)(0.1) = 0.06$
$T \to P \to P : (0.1)(0.8) = 0.08$
$T \to H \to T : (0.25)(0.09) = 0.0225$
$T \to H \to P : (0.25)(0.05) = 0.0125$
Total Prob. $= \boxed{0.535}$

(c) $H \to H \to H \to H : (0.85)(0.85)(0.85) = 0.614125$
$H \to T \to H \to H : (0.09)(0.25)(0.85) = 0.019125$
$H \to H \to T \to H : (0.85)(0.09)(0.25) = 0.019125$
$H \to T \to T \to H : (0.09)(0.6)(0.25) = 0.0135$
Total Prob. $= \boxed{0.665875}$

(d) $H \to H \to T \to D : (0.85)(0.09)(0.05) = 0.003825$
$H \to T \to H \to D : (0.09)(0.25)(0.01) = 0.000225$
$H \to T \to P \to D : (0.09)(0.1)(0.2) = 0.0018$
$H \to T \to T \to D : (0.09)(0.6)(0.05) = 0.0027$
$H \to T \to D \to D : (0.09)(0.05)(1) = 0.0045$
Total Prob. $= \boxed{0.01305}$

(e) (i) $H \to T \to D: PV = 100000/(1.1)^2 = 82644.63$;Prob $= (0.09)(0.05) = 0.0045$
$H \to P \to D: PV = 100000/(1.1)^2 = 82644.63$;Prob $= (0.05)(0.2) = 0.01$
$H \to H \to D: PV = 100000/(1.1)^2 = 82644.63$;Prob $= (0.85)(0.01) = 0.0085$
$H \to D \to D: PV = 100000/1.1 = 90909.09$;Prob $= 0.01$
$\boxed{\text{EPV=2809.92}}$

(ii) EPV(premiums) = EPV (expenses) + EPV(benefits)

$$\begin{aligned} G(1 + (0.99)/1.1) &= 0.05G(1 + (0.99)/1.1) + 2809.92 \\ 0.95G(1 + (0.99)/1.1) &= 2809.92 \\ G &= \frac{2809.92}{0.95(1 + (0.99)/1.1)} \\ G &= 1556.74 \end{aligned}$$

(iii) The gross premium reserve at time 1 is:
${}_1V^{(H)} = 0.01(100000)/1.1 - (0.95)1556.74 = -569.81$
${}_1V^{(T)} = 0.05(100000)/1.1 - (0.95)1556.74 = 3066.55$
${}_1V^{(P)} = 0.2(100000)/1.1 - (0.95)1556.74 = 16702.92$
${}_1V^{(D)} = 0$

(iv-i)
$H \to H \to P: PV = 10000/(1.1)^2 = 8264.46$; Prob $= (0.85)(0.05) = 0.0425$
$H \to H \to D: PV = 100000/(1.1)^2 = 82644.63$; Prob $= (0.85)(0.01) = 0.0085$
$H \to T \to P: PV = 10000/(1.1)^2 = 8264.46$; Prob $= (0.09)(0.1) = 0.009$
$H \to T \to D: PV = 100000/(1.1)^2 = 82644.63$; Prob $= (0.09)(0.05) = 0.0045$
$H \to P \to P: PV = 10000/(1.1) = 9090.91$; Prob $= (0.05)(0.8) = 0.04$
$H \to P \to D: PV = 10000\left(\frac{1}{1.1} + \frac{10}{(1.1)^2}\right) = 91735.54$; Prob $= (0.05)(0.2) = 0.01$
$H \to D \to D: PV = 100000/1.1 = 90909.09$; Prob $= 0.01$
$\boxed{\text{EPV} = 3690.08}$

(iv-ii) EPV(premiums) = EPV (expenses) + EPV(benefits)

$$\begin{aligned} G(1 + (0.85)/1.1) &= 0.05G(1 + (0.85)/1.1) + 3690.08 \\ 0.95G(1 + (0.85)/1.1) &= 3690.08 \\ G &= \frac{3690.08}{0.95(1 + (0.85)/1.1)} \\ G &= 2191.14 \end{aligned}$$

(iv-iii) The gross premium reserve at time 1 is:
${}_1V^{(H)} = [0.01(100000) + 0.05(10000)]/1.1 - (0.95)2191.14 = -717.95$

$${}_1V^{(T)} = [0.05(100000) + 0.1(10000)]/1.1 = 5454.55$$
$${}_1V^{(P)} = 0.2(100000)/1.1 = 18181.82$$
$${}_1V^{(D)} = 0$$

2. For the Kolmogorov Forward Equation Example in the class notes, fill in the remainder of the table. (Slide 1.13 and here)

Answer:

t	μ_{20+t}^{01}	μ_{20+t}^{02}	μ_{20+t}^{12}	μ_{20+t}^{10}	μ_{20+t}^{20}	μ_{20+t}^{21}	${}_tp_{20}^{00}$	${}_tp_{20}^{01}$	${}_tp_{20}^{02}$
0	0.03	0.06	0.1	0	0	0	1	0	0
0.25	0.03	0.0605	0.101	0	0	0	0.9775	0.0075	0.015
0.5	0.03	0.061	0.102	0	0	0	0.9554	0.0146	0.03
0.75	0.03	0.0615	0.103	0	0	0	0.9336	0.0214	0.0449
1	0.03	0.062	0.104	0	0	0	0.9123	0.0279	0.0598

3. We model the career path of an actuary using a continuous-time model with four states: Student (0), SOA member (1), CAS member (2), Retired (3). You are given the following forces of transition for an actuary age x:

- $\mu_x^{01} = 0.06$
- $\mu_x^{02} = 0.02$
- $\mu_x^{13} = 0.01 + \frac{x}{1000}$
- $\mu_x^{23} = a$

(a) Draw an appropriate state diagram for this model.

(b) Suppose that for someone age 45 who is currently a CAS member, the probability that they will be a retired by age 65 is 0.8. Calculate a. [0.08047]

(c) For a student who is currently age 20, calculate the probability they will still be a student at age 30. [0.4493]

(d) For a student who is currently age 25, write an expression for the probability that they will be an SOA member at age 45. Be as explicit as possible in your expression, i.e., give as detailed an expression as you can, but do not evaluate any integrals.

(e) For a member of the SOA who is currently 50, calculate the probability that they are still an SOA member at age 60. [0.52205]

(f) Consider the following expression:

$$\int_0^{30} \left[\int_0^t e^{-\int_0^s \mu_{20+r}^{01} + \mu_{20+r}^{02}\, dr} \cdot \mu_{20+s}^{02} \cdot e^{-\int_0^{t-s} \mu_{20+s+r}^{23}\, dr}\, ds \right] \cdot \mu_{20+t}^{23}\, dt$$

(i) Write a verbal explanation of the probability described by this expression.

(ii) Write a simpler expression (i.e., one involving a single integral) for this same probability.

(g) Suppose that CAS membership dues are payable continuously at a rate of \$500 per year. For a CAS member who is currently age 50, calculate the EPV of their future membership dues, assuming a force of interest of $\delta = 0.05$. [3832.29]

(h) Calculate the probability that a person who has just become a CAS member at age 40 pays more than \$5,000 in dues to the CAS in their lifetime. [0.44722]

Answer:

(a)

(b) First note that

$$ {}_tp_{45}^{\overline{22}} = \exp\left[-\int_0^t \mu_{45+s}^{23}\, ds\right] = \exp\left[-\int_0^t a\, ds\right] = e^{-at} $$

so that

$$
\begin{aligned}
{}_{20}p_{45}^{23} &= \int_0^{20} {}_tp_{45}^{\overline{22}}\, \mu_{45+t}^{23}\, dt \\
&= \int_0^{20} {}_tp_{45}^{\overline{22}}\, \mu_{45+t}^{23}\, dt \\
&= \int_0^{20} e^{-at}\, a\, dt \\
&= -e^{-at}\Big|_0^{20} \\
0.8 &= 1 - e^{-20a} \\
e^{-20a} &= 0.2
\end{aligned}
$$

$$\boxed{\text{a=0.08047}}$$

(c)

$$ {}_{10}p_{20}^{00} = {}_{10}p_{20}^{\overline{00}} = \exp\left\{-\int_0^{10} \mu_{20+s}^{01} + \mu_{20+s}^{02}\, ds\right\} = \exp\left\{-\int_0^{10} 0.08\, ds\right\} = e^{-0.8} = \boxed{0.4493} $$

(d)

$$
\begin{aligned}
{}_{20}p_{25}^{01} &= \int_0^{20} {}_tp_{25}^{\overline{00}} \cdot \mu_{25+t}^{01} \cdot {}_{20-t}p_{25+t}^{\overline{11}}\, dt \\
&= \int_0^{20} \exp\left[-\int_0^t \mu_{25+s}^{01} + \mu_{25+s}^{02}\, ds\right] \cdot \mu_{25+t}^{01} \cdot \exp\left[-\int_0^{20-t} \mu_{25+t+s}^{13}\, ds\right]\, dt \\
&= \int_0^{20} \exp\left[-\int_0^t 0.08\, ds\right] \cdot (0.06) \cdot \exp\left[-\int_0^{20-t} 0.01 + \frac{25+t+s}{1000}\, ds\right]\, dt
\end{aligned}
$$

(e)

$$
\begin{aligned}
{}_{10}p_{50}^{11} &= {}_{10}p_{50}^{\overline{11}} \\
&= \exp\left[-\int_0^{10} \mu_{50+s}^{13}\, ds\right] \\
&= \exp\left[-\int_0^{10} 0.01 + \frac{50+s}{1000}\, ds\right] \\
&= \exp\left[-\int_0^{10} 0.01 + 0.05 + 0.001s\, ds\right] \\
&= \exp\left[-\int_0^{10} 0.06 + 0.001s\, ds\right] \\
&= \exp\left[-0.06s - 0.0005s^2\Big|_0^{10}\right] \\
&= \exp\left[-0.65\right] = \boxed{0.52205}
\end{aligned}
$$

(f) (i) This is the probability that a student age 20 is retired by age 50, having been a member of the CAS at some point in his career.

(ii)

$$
\int_0^{30} {}_tp_{20}^{02}\, \mu_{20+t}^{23}\, dt
$$

(g) The EPV of their dues would be

$$
\begin{aligned}
\int_0^{\infty} 500\, e^{-\delta t}\, {}_tp_{50}^{\overline{22}}\, dt &= \int_0^{\infty} 500\, e^{-\delta t}\, e^{-at}\, dt \\
&= \int_0^{\infty} 500\, e^{-(\delta+a)\,t}\, dt \\
&= \frac{-500}{\delta+a}\, e^{-(\delta+a)\,t}\Big|_0^{\infty} \\
&= \frac{500}{\delta+a} \\
&= \frac{500}{0.05+0.08047} \\
&= \boxed{3832.29}
\end{aligned}
$$

(h) The person will pay more than \$5,000 in dues to the CAS if they remain a member for at least 10 years, which is

$$
{}_{10}p_{40}^{22} = \exp\left\{-\int_0^{10} a\, dt\right\} = e^{-10a} = e^{0.8047} = \boxed{0.44722}
$$

Part II

1. We model the employment status of a person initially age y (where, as usual, y is the last two digits of your student ID number) with a three state model: Employed (0), Unemployed (1), and Dead (2). The forces of transition (for a person at **attained age x**) are:[1]

   - $\mu_x^{01} = 0.2 + 0.0002x^2$
   - $\mu_x^{02} = 0.05$
   - $\mu_x^{12} = 0.06$
   - $\mu_x^{10} = 0.08$
   - $\mu_x^{20} = 0$
   - $\mu_x^{21} = 0$

   We want to use this model in order to price a 20 year unemployment insurance that pays 1400 at the beginning of each month that the person is unemployed. (The first potential payment is at time $\frac{1}{12}$ and the last potential payment is at time 20.) We will assume that the insurance is sold to people age y who are currently employed and that $i^{(12)} = 0.04$.

   (a) Using Euler's Forward Method with step size $h = \frac{1}{12}$, calculate ${}_tp_y^{00}, {}_tp_y^{01},$ and ${}_tp_y^{02}$ for $t = \frac{1}{12}, \frac{2}{12}, \ldots, 20$.

   (b) Find the EPV at issue of this insurance.

   (c) Now suppose that the insured is required to pay a level premium at the beginning of each month that he is Employed. (The first premium is at time 0 and the last potential premium is at time $20 - \frac{1}{12}$.) Find the amount of this premium, according to the equivalence principle.

   (d) Now use the same model to find the net single premium for a 20 year term life insurance issued to a person age y which pays a death benefit of \$500,000 at the end of the month of death. (Hint: Note that the insurance will pay only at the <u>first</u> time period the Dead state is entered.)

[1] Example: If your BYU student ID is 123-12-3456, then $y = 56$. For this person, at time 0, $\mu_{56}^{01} = 0.2 + 0.0002(56^2)$, and at time 2, $\mu_{58}^{01} = 0.2 + 0.0002(58^2)$.