Stat 444 — Winter 2024

Homework Assignment 3

Due Date: Thursday, February 29th at 2:00 pm, but the material is on the midterm exam

General Notes:

- For Part I, you may submit your assignment to Learning Suite or in class.
- For Part II, you should use a spreadsheet.
  - Submit electronically the Excel spreadsheet you create to answer the questions in Part II to Learning Suite. Your spreadsheet should be neatly organized and labeled; each answer should be highlighted in some manner, and it should be very clear how each of your answers was obtained.

Part I

Problems 1 - 3 refer to the following:

You are given the following information regarding the joint life multi-state model:

$$\mu_{xy}^{01} = \mu_{y}^{23} = 0.04 \qquad \mu_{xy}^{02} = \mu_{x}^{13} = 0.06 \qquad \mu_{xy}^{03} = 0.01 \qquad \delta = 0.1$$

1. Calculate the following probabilities and give the symbols for each probability (in traditional actuarial notation):
   (a) The probability that (x) dies before (y). [0.545]
   (b) The probability that both (x) and (y) are still alive in 10 years. [0.33287]
   (c) The probability that both (x) and (y) are dead in 10 years. [0.2051345]

Answer: First, we'll calculate some probabilities that will be used in these problems:

$$
\begin{aligned}
{}_tp_{xy}^{00} &= e^{-\int_0^t 0.11\,ds} \\
&= e^{-0.11t} \\
{}_tp_{xy}^{11} &= e^{-0.06t} \\
{}_tp_{xy}^{22} &= e^{-0.04t} \\
{}_tp_{xy}^{01} &= \int_0^t {}_sp_{xy}^{00}\,\mu_{x+s:y+s}^{01}\,{}_{t-s}p_{x+s:y+s}^{11}\,ds \\
&= \int_0^t e^{-0.11s}\,(0.04)\,e^{-0.06(t-s)}\,ds \\
&= (0.04)e^{-0.06t}\int_0^t e^{-0.05s}\,ds \\
&= (0.8)e^{-0.06t}\left[1-e^{-0.05t}\right] \\
{}_tp_{xy}^{02} &= \int_0^t {}_sp_{xy}^{00}\,\mu_{x+s:y+s}^{02}\,{}_{t-s}p_{x+s:y+s}^{22}\,ds \\
&= \int_0^t e^{-0.11s}\,(0.06)\,e^{-0.04(t-s)}\,ds \\
&= (0.06)e^{-0.04t}\int_0^t e^{-0.07s}\,ds \\
&= (0.8571)e^{-0.04t}\left[1-e^{-0.07t}\right]
\end{aligned}
$$

(a)

$$
\begin{aligned}
{}_\infty q_{\overset{1}{x}y} &= \int_0^\infty {}_tp_{xy}^{00}\,\mu_{x+t:y+t}^{02}\,dt \\
&= \int_0^\infty e^{-0.11t}\,(0.06)\,dt \\
&= \boxed{6/11} \text{ or } \boxed{0.545}
\end{aligned}
$$

(b)

$$
\begin{aligned}
{}_{10}p_{xy} &= {}_{10}p_{xy}^{00} \\
&= \int_0^{10} e^{-0.11\,dt} \\
&= e^{-1.1} \\
&= \boxed{0.33287}
\end{aligned}
$$

(c)

$$
\begin{aligned}
{}_{10}q_{\overline{xy}} &= 1 - {}_{10}p_{xy}^{00} - {}_{10}p_{xy}^{01} - {}_{10}p_{xy}^{02} \\
&= \boxed{0.2051345}
\end{aligned}
$$

2. Calculate the following EPV values:

   (a) $\overline{A}_x$ [0.404762]

   (b) $\overline{A}_{xy}$ [0.5238]

   (c) $A_{\overset{1}{y}:\overline{1|}}$ [0.043875]

   (d) $\overline{a}_{x:y:\overline{5|}}$ [3.0955]

   (e) $\overline{a}_{\overline{xy}}$ [7.993095]

   (f) $\overline{a}_{x|y}$ [2.0408]

   (g) $\ddot{a}_{\overline{x:y:\overline{2|}}:\overline{4|}}$ [3.410018]

   Answer:

   (a)

$$
\begin{aligned}
\overline{A}_x &= \int_0^\infty e^{-\delta\, t}\,\left({}_tp_{xy}^{00}\,\mu_{x+t:y+t}^{02} + {}_tp_{xy}^{00}\,\mu_{x+t:y+t}^{03} + {}_tp_{xy}^{01}\,\mu_{x+t:y+t}^{13}\right)\,dt \\
&= \int_0^\infty e^{-\delta\, t}\,\left(e^{-0.11t}\,(0.06) + e^{-0.11t}\,(0.01) + (0.8)e^{-0.06t}\left[1 - e^{-0.05t}\right]\,(0.06)\right)\,dt \\
&= \int_0^\infty e^{-\delta\, t}\,\left(e^{-0.11t}\,(0.07) + (0.8)e^{-0.06t}\left[1 - e^{-0.05t}\right]\,(0.06)\right)\,dt \\
&= \int_0^\infty e^{-\delta\, t}\,\left(e^{-0.11t}\,(0.07) + (0.048)e^{-0.06t} - (0.048)e^{-0.06t}e^{-0.05t}\right)\,dt \\
&= \int_0^\infty e^{-\delta\, t}\,\left(e^{-0.11t}\,(0.022) + (0.048)e^{-0.06t}\right)\,dt \\
&= \int_0^\infty e^{-0.21t}\,(0.022) + (0.048)e^{-0.16t}\,dt \\
&= \boxed{0.404762}
\end{aligned}
$$

   (b)

$$
\begin{aligned}
\overline{A}_{xy} &= \int_0^\infty e^{-\delta\, t}\,{}_tp_{xy}^{00}\,\left(\mu_{x+t:y+t}^{01} + \mu_{x+t:y+t}^{02} + \mu_{x+t:y+t}^{03}\right)\,dt \\
&= \int_0^\infty e^{-0.21t}(0.11)\,dt \\
&= \frac{0.11}{0.21} \\
&= \boxed{0.5238}
\end{aligned}
$$

(c)

$$
\begin{aligned}
A_{\substack{1\\ y:\overline{1}|}} &= e^{-\delta}\,(1-p_y) \\
&= e^{-\delta}\left(1-p_{xy}^{00}-p_{xy}^{02}\right) \\
&= e^{-0.1t}\left(1-e^{-0.11t}-(0.8571)e^{-0.04t}\left[1-e^{-0.07t}\right]\right) \\
&= \boxed{0.043875}
\end{aligned}
$$

(d)

$$
\begin{aligned}
\overline{a}_{x:y:\overline{5}|} &= \int_0^5 e^{-\delta t}\,{}_tp_{xy}^{00}\,dt \\
&= \int_0^5 e^{-0.21t}\,dt \\
&= \boxed{3.0955}
\end{aligned}
$$

(e)

$$
\begin{aligned}
\overline{a}_{\overline{xy}} &= \int_0^\infty e^{-\delta t}\,\left({}_tp_{xy}^{00}+{}_tp_{xy}^{01}+{}_tp_{xy}^{02}\right)\,dt \\
&= \int_0^\infty e^{-\delta t}\,\left(e^{-0.11t}+(0.8)e^{-0.06t}\left[1-e^{-0.05t}\right]+(0.8571)e^{-0.04t}\left[1-e^{-0.07t}\right]\right)\,dt \\
&= \int_0^\infty e^{-0.1t}\,\left(e^{-0.11t}+(0.8)e^{-0.06t}\left[1-e^{-0.05t}\right]+(0.8571)e^{-0.04t}\left[1-e^{-0.07t}\right]\right)\,dt \\
&= \int_0^\infty \left(e^{-0.21t}+(0.8)e^{-0.16t}\left[1-e^{-0.05t}\right]+(0.8571)e^{-0.14t}\left[1-e^{-0.07t}\right]\right)\,dt \\
&= \int_0^\infty \left(0.8e^{-0.16t}+0.8571e^{-0.14t}-0.6571e^{-0.21t}\right)\,dt \\
&= \boxed{7.993095}
\end{aligned}
$$

(f)

$$
\begin{aligned}
\overline{a}_{x|y} &= \int_0^\infty e^{-\delta t}\,{}_tp_{xy}^{02}\,dt \\
&= \int_0^\infty e^{-0.1t}\,(0.8571)e^{-0.04t}\left[1-e^{-0.07t}\right]\,dt \\
&= (0.8571)\int_0^\infty e^{-0.14t}-e^{-0.21t}\,dt \\
&= \boxed{2.0408}
\end{aligned}
$$

(g)

$$
\begin{aligned}
\ddot{a}_{\overline{x:y:\overline{2|}}:\overline{4|}} &= 1 + e^{-\delta} + e^{-2\delta}\left({}_2p_{xy}^{00} + {}_2p_{xy}^{01} + {}_2p_{xy}^{02}\right) + e^{-3\delta}\left({}_3p_{xy}^{00} + {}_3p_{xy}^{01} + {}_3p_{xy}^{02}\right) \\
&= \boxed{3.410018}
\end{aligned}
$$

3. Let Y be the random variable representing the PV of a whole life last survivor annuity paying continuously at a rate of \$10,000 per year issued to (x) and (y).

   (a) Calculate $P(Y > 50000)$. [0.870828]

   (b) Verify that $P\left(T_{\overline{xy}} > 75.63\right) = 0.05$.

   (c) The insurer charges a single premium for this product such that the probability that they lose money on this annuity is no more than 5%. Find the single premium for this product. [99948.07]

Answer:

(a)

$$
\begin{aligned}
P(Y > 50000) &= P\left(10000\left[\frac{1 - e^{-\delta T_{\overline{xy}}}}{\delta}\right] > 50000\right) \\
&= P\left(\frac{1 - e^{-\delta T_{\overline{xy}}}}{\delta} > 5\right) \\
&= P\left(1 - e^{-\delta T_{\overline{xy}}} > 5\delta\right) \\
&= P\left(1 - 5\delta > e^{-\delta T_{\overline{xy}}}\right) \\
&= P\left(\ln(1 - 5\delta) > -\delta T_{\overline{xy}}\right) \\
&= P\left(-\ln(1 - 5\delta) < \delta T_{\overline{xy}}\right) \\
&= P\left(T_{\overline{xy}} > 6.93147\right) \\
&= {}_{6.93147}p_{xy}^{00} + {}_{6.93147}p_{xy}^{01} + {}_{6.93147}p_{xy}^{02} \\
&= \boxed{0.870828}
\end{aligned}
$$

(b)

$$
\begin{aligned}
P\left(T_{\overline{xy}} > 75.63\right) &= {}_{75.63}p_{xy}^{00} + {}_{75.63}p_{xy}^{01} + {}_{75.63}p_{xy}^{02} \\
&= e^{-0.11(75.63)} + (0.8)e^{-0.06(75.63)}\left[1 - e^{-0.05(75.63)}\right] \\
&\qquad + (0.8571)e^{-0.04(75.63)}\left[1 - e^{-0.07(75.63)}\right] \\
&= 0.05
\end{aligned}
$$

(c) Let P be the net single premium.

$$\begin{aligned} 0.05 &= P(Y > P) \\ &= P\left(10000\left[\frac{1-e^{-\delta T_{\overline{xy}}}}{\delta}\right] > P\right) \\ &= P\left(T_{\overline{xy}} > \frac{-\ln(1-\delta P/10000)}{\delta}\right) \end{aligned}$$

Then

$$\begin{aligned} \frac{-\ln(1-\delta P/10000)}{\delta} &= 75.63 \\ P &= 100000\left(1-e^{-7.563}\right) \\ P &= \boxed{99948.07} \end{aligned}$$

4. Jack and Jill, who are ages 50 and 60 respectively and have independent future lifetimes whose mortality follows the SULT, purchase a fully discrete 20 year term insurance policy. The policy pays a benefit of $250,000 upon the second death. Annual premiums are payable as long as both Jack and Jill are alive. Use $i = 5\%$.

(a) Calculate the net annual premium for this policy. [153.04]

(b) Calculate the probability that this policy pays a benefit. [0.016503]

(c) Calculate the net premium reserve at time 10, under the assumption that:

(i) Only Jack is living. [10630]

(ii) Only Jill is living. [31455]

(iii) Jack and Jill are both living. [513.02]

Answer:

(a) First, we'll calculate a few quantities we'll need:

$$ {}_{20}E_{50:60} \overset{indep}{=} {}_{20}E_{50} \cdot {}_{20}E_{60} \cdot (1.05)^{20} = (0.34824)(0.29508)(1.05)^{20} = 0.272649$$

$$\ddot{a}_{50:60:\overline{20|}} = \ddot{a}_{50:60} - {}_{20}E_{50:60}\,\ddot{a}_{70:80} = 14.2699 - (0.272649)(7.7208) = 12.16483$$

$$A^{\ 1}_{50:\overline{20|}} = A_{50:\overline{20|}} - {}_{20}E_{50} = 0.38844 - 0.34824 = 0.0402$$

$$A^{\ 1}_{60:\overline{20|}} = A_{60:\overline{20|}} - {}_{20}E_{60} = 0.41040 - 0.29508 = 0.11532$$

$$A^{\ \ 1}_{\underbrace{50:60}:\overline{20|}} = A_{50:60} - {}_{20}E_{50:60}\,A_{70:80} = 0.32048 - (0.272649)(0.63234) = 0.148073$$

Then we can calculate the premium as:

$$\begin{aligned}
P\ddot{a}_{50:60:\overline{20|}} &= 250000 A_{\overset{1}{50:60}:\overline{20|}} \\
P(12.16483) &= 250000 \left(A_{\overset{1}{50}:\overline{20|}} + A_{\overset{1}{60}:\overline{20|}} - A_{\overbrace{50:60}^{1}:\overline{20|}} \right) \\
P(12.16483) &= 250000(0.0402 + 0.11532 - 0.148073) \\
P &= \boxed{153.04}
\end{aligned}$$

(b) The policy pays a benefit if both insureds die within 20 years:

$$\begin{aligned}
{}_{20}q_{\overline{xy}} &\overset{indep.}{=} {}_{20}q_x \cdot {}_{20}q_y \\
&= \left(1 - \frac{\ell_{70}}{\ell_{50}}\right) \cdot \left(1 - \frac{\ell_{80}}{\ell_{60}}\right) \\
&= \left(1 - \frac{91082.4}{98576.4}\right) \cdot \left(1 - \frac{75657.2}{96634.1}\right) \\
&= \boxed{0.016503}
\end{aligned}$$

(c) Again, we'll first calculate a few needed quantities:

$$\begin{aligned}
{}_{10}E_{60:70} &\overset{indep}{=} {}_{10}E_{60} \cdot {}_{10}E_{70} \cdot (1.05)^{10} = (0.57864)(0.50994)(1.05)^{10} = 0.48064 \\
A_{\overset{1}{60}:\overline{10|}} &= A_{60:\overline{10|}} - {}_{10}E_{60} = 0.62116 - 0.57864 = 0.04252 \\
A_{\overset{1}{70}:\overline{10|}} &= A_{70:\overline{10|}} - {}_{10}E_{70} = 0.63576 - 0.50994 = 0.12582 \\
A_{\overbrace{60:70}^{1}:\overline{10|}} &= A_{60:70} - {}_{10}E_{60:70}\, A_{70:80} = 0.46562 - (0.48064)(0.63234) = 0.16169
\end{aligned}$$

(i)

$${}_{10}V = 250000 A^{1}_{60:\overline{10|}} = 250000(0.04252) = \boxed{10630}$$

(ii)

$${}_{10}V = 250000 A^{1}_{70:\overline{10|}} = 250000(0.12582) = \boxed{31455}$$

(iii)

$$\begin{aligned}
{}_{10}V &= 250000 A_{\overset{1}{60:70}:\overline{10|}} - P\,\ddot{a}_{60:70:\overline{10|}} \\
&= 250000\left(A_{\overset{1}{60}:\overline{10|}} + A_{\overset{1}{70}:\overline{10|}} - A_{\overset{1}{\overbrace{60:70}}:\overline{10|}}\right) - (153.04)\ddot{a}_{60:70:\overline{10|}} \\
&= 250000\,(0.04252 + 0.12582 - 0.16169) - (153.04)(7.5110) \\
&= \boxed{513.02}
\end{aligned}$$

5. Find the value of $p_x'^{(1)}$, given $q_x^{(1)} = 0.48$, $q_x^{(2)} = 0.32$, $q_x^{(3)} = 0.16$, and each decrement is uniformly distributed over $(x, x+1)$ in the multiple-decrement context. [0.2]

   Answer:
$$p_x'^{(1)} = \left(1 - q_x^{(\tau)}\right)^{q_x^{(1)}/q_x^{(\tau)}} = (0.04)^{0.48/0.96} = \boxed{0.2}$$

6. Find the value of ${}_{0.5}p_x'^{(1)}$, given $q_x^{(1)} = 0.48$, $q_x^{(2)} = 0.32$, $q_x^{(3)} = 0.16$, and each force of decrement is constant within the year. [0.4472136]

   Answer:
$${}_{0.5}p_x'^{(1)} = \left([0.04]^{0.48/0.96}\right)^{0.5} = \boxed{0.4472136}$$

7. Find the value of $q_x^{(1)}$, given $q_x'^{(1)} = 0.2$, $q_x'^{(2)} = 0.1$, and decrements are uniformly distributed over $(x, x+1)$ in the multiple-decrement context. [0.190196]

   Answer: $p_x'^{(1)} = 0.8$, $p_x'^{(2)} = 0.9 \Rightarrow p_x^{(\tau)} = 0.72 \Rightarrow q_x^{(\tau)} = 0.28$.

$$\begin{aligned}
p_x'^{(1)} &= \left(1 - q_x^{(\tau)}\right)^{q_x^{(1)}/q_x^{(\tau)}} \\
0.8 &= (0.72)^{q_x^{(1)}/0.28} \\
\ln(0.8) &= (q_x^{(1)}/0.28)\ln(0.72) \\
q_x^{(1)} &= \frac{\ln(0.8)\cdot(0.28)}{\ln(0.72)} \\
q_x^{(1)} &= \boxed{0.190196}
\end{aligned}$$

8. Decrement 1 is uniformly distributed over the year of age in its associated single decrement table with $q_x'^{(1)} = 0.1$. Decrement 2 always occurs at age $x + 0.7$ in its associated single decrement table with $q_x'^{(2)} = 0.125$. Find the value of $q_x^{(2)}$. [0.11625]

   Answer: Imagining that there are 1000 people in the population at age x, then decrement 1 alone will act on the population for the first 0.7 years, so that after this time there will be $1000(1 - 0.7 * 0.1) = 930$ left at time 0.7. Then decrement 2 acts and removes 12.5% of the remaining population, which is $0.125 * 930 = 116.25$. Then $q_x^{(2)} = 116.25/1000 = \boxed{0.11625}$.

9. Students can leave a certain three-year school only for reasons of failure (1) or voluntary withdrawal (2), where each decrement is uniformly distributed over $(x, x+1)$ in its associated single-decrement table. The following values are given:

x	$q_x^{\prime(1)}$	$q_x^{\prime(2)}$	$q_x^{(1)}$	$q_x^{(2)}$
0	0.10	0.25	$---$	$---$
1	0.20	0.20	$---$	$---$
2	0.20	0.10	$---$	$---$

(a) Calculate the six missing probability values for the table. [0.0875, 0.2375, 0.18, 0.18, 0.19, 0.09]

(b) Given that a person decrements from school in the third year, find the probability that the decrement was a failure. [0.67857]

(c) Given that a student enters the second year, find the probability of eventually decrementing due to failure. [0.3016]

Answer:

(a) Use the formulas

$$q_x^{(1)} = q_x^{\prime(1)}\left(1 - \frac{1}{2}\,q_x^{\prime(2)}\right) \text{ and } q_x^{(2)} = q_x^{\prime(2)}\left(1 - \frac{1}{2}\,q_x^{\prime(1)}\right)$$

to get the table below:

x	$q_x^{\prime(1)}$	$q_x^{\prime(2)}$	$q_x^{(1)}$	$q_x^{(2)}$
0	0.10	0.25	0.0875	0.2375
1	0.20	0.20	0.18	0.18
2	0.20	0.10	0.19	0.09

(b)

$$\frac{q_2^{(1)}}{q_2^{(\tau)}} = \frac{0.19}{0.28} = \boxed{0.67857}$$

(c)

$$0.18 + 0.64(0.19) = \boxed{0.3016}$$

10. Suppose that you have a triple decrement table. Assuming that decrements are uniformly distributed in the associated single decrement tables,

(a) Show that

$$q_x^{(1)} = q_x^{\prime(1)}\left[1 - \frac{1}{2}\left(q_x^{\prime(2)} + q_x^{\prime(3)}\right) + \frac{1}{3}\left(q_x^{\prime(2)} \cdot q_x^{\prime(3)}\right)\right]$$

(b) Derive (or just write down) formulas for $q_x^{(2)}$ and $q_x^{(3)}$ under this fractional age assumption.

Answer:

(a)

$$
\begin{aligned}
q_x^{(1)} &= \int_0^1 {}_tp_x^{(\tau)}\, \mu_{x+t}^{(1)}\, dt \\
q_x^{(1)} &= \int_0^1 {}_tp_x'^{(1)}\, {}_tp_x'^{(2)}\, {}_tp_x'^{(3)}\, \mu_{x+t}^{(1)}\, dt \\
q_x^{(1)} &= \int_0^1 (1 - t\cdot q_x'^{(1)})\, {}_tp_x'^{(2)}\, {}_tp_x'^{(3)}\, \frac{q_x'^{(1)}}{1 - t\cdot q_x'^{(1)}}\, dt \\
q_x^{(1)} &= \int_0^1 {}_tp_x'^{(2)}\, {}_tp_x'^{(3)}\, q_x'^{(1)}\, dt \\
q_x^{(1)} &= q_x'^{(1)} \int_0^1 (1 - t\cdot q_x'^{(2)})\,(1 - t\cdot q_x'^{(3)})\, dt \\
q_x^{(1)} &= q_x'^{(1)} \int_0^1 \left[1 - t(q_x'^{(2)} + q_x'^{(3)}) + t^2\cdot q_x'^{(2)}\cdot q_x'^{(3)}\right]\, dt \\
q_x^{(1)} &= q_x'^{(1)} \left[t - \frac{1}{2}t^2(q_x'^{(2)} + q_x'^{(3)}) + \frac{1}{3}t^3\cdot q_x'^{(2)}\cdot q_x'^{(3)}\right]_0^1 \\
q_x^{(1)} &= q_x'^{(1)} \left[1 - \frac{1}{2}(q_x'^{(2)} + q_x'^{(3)}) + \frac{1}{3}\cdot q_x'^{(2)}\cdot q_x'^{(3)}\right]
\end{aligned}
$$

(b)

$$
\begin{aligned}
q_x^{(2)} &= q_x'^{(2)} \left[1 - \frac{1}{2}\left(q_x'^{(1)} + q_x'^{(3)}\right) + \frac{1}{3}\left(q_x'^{(1)}\cdot q_x'^{(3)}\right)\right] \\
q_x^{(3)} &= q_x'^{(3)} \left[1 - \frac{1}{2}\left(q_x'^{(1)} + q_x'^{(2)}\right) + \frac{1}{3}\left(q_x'^{(1)}\cdot q_x'^{(2)}\right)\right]
\end{aligned}
$$

Part II

For this part, you'll be modeling the future lifetimes of two people, (x) and (y), who have independent future lifetimes, using the Standard Ultimate Mortality Model. As usual, y is the last two digits of your BYU ID number; add 20 if $y < 20$. Let $x = y + 5$. Assume that $i = 4\%$.

1. Create a column with the pmf of K_{xy} and create a graph that shows the pmf.

2. Calculate e_{xy}.

3. Calculate the EPV of a \$500,000 fully discrete 20-year first-to-die (joint) term policy.

4. Assuming that premiums are payable while both (x) and (y) are alive, calculate the annual premium for the policy above.

5. Calculate the probability that this policy will pay a benefit.