Stat 444
Advanced Long-term Actuarial Math

Lecture 7: Embedded Options

Variable Annuities

Variable annuities (VAs) are equity-linked accumulation products[1] that provide the policyholder tax-deferred growth on investments.

- They are often used as retirement savings vehicles.
- The first VA product was introduced by TIAA-CREF in 1952, but didn't explode in popularity until the 1990s.
- According to LIMRA, in 2022, the U.S. saw \$310.6 billion in annuity sales, of which \$61.7 billion was due to VAs.
  - Annuity sales in general increased significantly from 2021, but VA sales decreased; both fixed annuities and indexed annuities increased.

[1] There are also variable payout (immediate) annuities whose payments fluctuate with some index, fund, or market, but we're focusing here on the deferred variety.

Variable Annuity Mechanics

When a policyholder purchases a VA product, they deposit money with the LIC.

- Like a VUL product, the policyholder can choose how the money is allocated among various funds.
- These funds operate like mutual funds (i.e., with unit values that fluctuate with the market).
- The choices of funds is similar to that of a VUL product.
- Throughout the **accumulation phase**, deposits (premiums) are flexible, and the allocation of funds is directed by the policyholder and can be changed at their discretion.

Variable Annuity Mechanics

At some specified age, the policyholder can convert the accumulated account balance into a stream of payments lasting for their lifetime, i.e., begin the **payout phase**.

- The annuitization factors ($\ddot{a}_x$) may be specified in the contract; payout can be fixed or variable.
- Other annuitization options (first-to-die, last-to-die, life-and-n-year-certain, etc.) are often also available.
- The other option is to surrender the product.
  - There is usually a decreasing surrender charge scale that goes to zero after, e.g., 3 to 10 years.
- If the policyholder dies during the accumulation phase, a death benefit is payable.

Variable Annuity Tax Benefits

One of the appeals of VA products is their favorable tax treatment.

- Variable annuities are tax-deferred, so that no taxes on the income and investment gains are due until the money is withdrawn.
- Internal transfers (i.e., moving money from one investment option to another) don't incur any taxes, either.
- Money withdrawn from a VA incurs taxes at ordinary income tax rates. Money withdrawn prior to age 59 $1/2$ incurs an additional 10% federal tax, similar to a withdrawal from a 401(k) plan.

Another one of the appeals of VA products is the various guarantees they almost always include.

- These features provide minimum guaranteed values for various quantities, offering the policyholder a measure of downside protection.
- These guarantees are commonly known collectively as **GMxB**.

Notation, Terminology, and Assumptions

Some notation, terminology, and simplifying assumptions:

- The **policyholder's fund** F_t is the policyholder's account balance at time t.
- We assume a single **premium** P payable at issue (time 0). (Further premium contributions are almost always allowable in practice.)
- The **maturity date** or **term** of the policy is denoted by n. (This marks the end of the accumulation phase.)
- The income-related guarantees (GMIB, GMWB) are based on the **benefit base**, BB_t, which can be defined in various ways.
- The **management charge** MC_t is the amount deducted from the policyholder's fund at time t. (Usually some percentage of F_t and/or BB_t.)
- The **surrender charge** at time t is given by SC_t.

VA Guarantees: GMDB and GMMB

A couple of the simpler common VA guarantees are:

- A **Guaranteed Minimum Death Benefit (GMDB)** guarantees that the death benefit (which is otherwise usually equal to the account balance or premium paid) is at least equal to some minimum guaranteed value.
- A **Guaranteed Minimum Maturity Benefit (GMMB)** guarantees that the accumulated account value will at least equal some minimum guaranteed value at maturity.

The simplest design is to have the guaranteed amount equal to some fixed percentage of the premium paid. But there are commonly-used mechanisms by which this guaranteed amount could increase over the life of the contract.

GMDB and GMMB Notation

We let G_t^M be the level of guaranteed maturity benefit that would be available if the policy matured at t. (But which is not actually available to the policyholder until time n.)

Similarly, G_t^D is the level of the guaranteed death benefit payable at time t.

The **cash value** (or **cash surrender value**) CV_t denotes the cash value of the contract at time t.

$$CV_t = \begin{cases} F_t - SC_t & t < n \\ \max(F_n, G_n^M) & t = n \end{cases}$$

Reset Option

A **reset option** allows the policyholder to reset the guarantee amount from time to time to match the fund value at that time. (If the reset occurs within the final k years of the contract, the maturity date is extended accordingly.)

Providing this option can benefit the LIC:

- It helps reduce **lapse-and-re-entry risk**, and makes policyholders whose account value has greatly increased less likely to surrender their policies and take their money elsewhere.
- Could be beneficial from a marketing perspective.

However, after a sudden market rise, many policyholders could reset around the same time, causing a concentration risk.

Step-up Guarantees

Under a **step-up guarantee**, the insurer will specify step-up dates throughout the term of the contract (e.g., each year or every five years).

At a step-up date the guarantee amount is increased to the level of the policyholder's fund at that date, if that is larger than the guarantee amount immediately before the step-up date.

- Thus, the guarantee amount at any point is the highest value that the policyholder's fund attained in each of the previous step-up dates.

Guaranteed Rollover

The **guaranteed rollover** is similar to a step-up guarantee, in that at specified dates, the guarantee amount is increased to the level of the policyholder's fund at that date, if that is larger than the guarantee amount immediately before the step-up date.

- However, if the fund is less than the current guarantee amount at that date, the LIC will increase the policyholder's fund up to the amount of the guarantee amount.

In practice, other guarantee amount increase patterns are seen. For example, a "roll-up" design may have the guarantee amount grow by at least a minimum percentage (e.g., 1% - 3%) each year.

Guaranteed Minimum Income Benefit (GMIB)

The **Guaranteed Minimum Income Benefit (GMIB)** provides a guaranteed annuitization rate for the benefit base at maturity.

- If the market annuitization rates at retirement are less favorable (i.e., more expensive) than the rate specified in the contract, then the policyholder can exercise their GMIB option to annuitize at the guaranteed rate.
- If market annuitization rates are cheaper than the guaranteed rate, then the policyholder can withdraw their funds and annuitize at the market rate, with the same insurer, or with a different provider.

Thus, the GMIB guarantees a minimum income stream upon annuitization, based on the specified annuity factors and the benefit base.

GMIB Notation, Assumptions, and Equations

We'll assume for simplicity that all annuities are plain whole life annuities-due.

We denote the guaranteed annuitization rate for a annuitant whose contract matures at age y by $\gamma_y^g = 1/\ddot{a}_y^g$.

Similarly, we denote the market annuitization rate at time t for a person age y by $\gamma_y(t) = 1/\ddot{a}_y(t)$.

More GMIB Notation, Assumptions, and Equations

Suppose the person whose contract matures at time n (when they're age y) wants to receive an income stream of B per year. Their cash value at maturity is

$$CV_n = max(F_n, G_n^M)$$

They could cash out their contract and purchase an annuity at current market rates to get an annual income of

$$B^{mkt} = CV_n \times \gamma_y(n)$$

Or they could exercise the GMIB option to get an annual income of

$$B^{GMIB} = BB_n \times \gamma_y^g$$

Guaranteed Minimum Withdrawal Benefit (GMWB)

Under a **Guaranteed Minimum Withdrawal Benefit (GMWB)**, after some specified waiting period, a specified minimum percentage of the benefit base is guaranteed to be able to be withdrawn, each year and/or over the life of the individual, even if their account runs out of funds.

- For example, a GMWB rider may provide for withdrawals starting at time n, of $c\%$ of BB_n each year. Typically, c ranges from 4% to 7%.
- The guarantee amount is usually fixed once the withdrawal period starts, as long as the policyholder does not withdraw more than the minimum amount.
- At any stage, provided $F_t > 0$, the policyholder may surrender and withdraw the remaining funds. If the policyholder dies, then any remaining funds belong to their estate.

VA Deterministic Profit Testing

If we want to perform a (deterministic) profit test for a VA product, we can do so similarly to a UL product:

- First we project forward the policyholder's fund, then determine the cash inflows and outflows for the insurer.
- Then we can calculate the profit vector, profit signature, and finally any desired profit measures.

For a product like VA, deterministic profit testing doesn't give a complete picture of profitability, but it can be useful for stress testing, i.e., calculating results under specific scenarios of interest.

A more complete approach is to use a stochastic methodology, such as Monte Carlo simulation, to generate a large number of random scenarios.

- We'd use some specified distribution (e.g., lognormal) to generate investment returns, then use these returns to calculate all of the needed cash flows for each scenario.
- We could then analyze the *distribution* of profit measures, such as NPV, to get a median, mean, 95^{th} percentile, etc.
- We could incorporate more realism by making other elements, such as policyholder behavior, dependent on the scenario returns.

VA Stochastic Reserving

We can use similar ideas to do reserving or valuation in a stochastic manner.

We might run a large number of scenarios, and then set the reserve to the $1 - \alpha$ percentile of the loss-at-issue RV. This idea is sometimes referred to as **Value at Risk (VaR)**.

Or we could use the mean of the worst α proportion of scenarios; this is called **Conditional Tail Expectation (CTE)**, or **Tail Value at Risk (TVaR)**.

However, this type of approach may not be ideal in the sense that it could require very large reserves which may rarely be required, lowering the profitability of the product.

Arbitrage and Replicating Portfolios

Option pricing (and much of the theory behind sophisticated mathematical finance) is based on the **no arbitrage assumption**, which states that two portfolios producing the same cash flows must have the same price.

This idea is often used to price options and other securities by constructing a **replicating portfolio** whose price can be more easily determined.

Notation and Terminology

A **European call option** allows its holder the right to buy a specified quantity of a security at a set price on some specified date.

Conversely, a **European put option** allows its holder the right to sell a specified quantity of a security at a set price on some specified date.

- The set price is called the **strike price** or **exercise price**, denoted by K.
- The date of the purchase or sale is called the **expiry date** or maturity date, often denoted by T.
- The price of the underlying security at time t is denoted S_t.

Option Payoffs

The payoff at maturity for a European call option is

$$(S_T - K)^+ = \max(S_T - K, 0)$$

The payoff at maturity for a European put option is

$$(K - S_T)^+ = \max(K - S_T, 0)$$

An option with a positive payoff is said to be "in the money", while options whose exercise would not produce a payoff are called "out of the money".

Typical Option Assumptions

We make a number of assumptions necessary or helpful to enable the construction of replicating portfolios.

- There is a risk-free asset earning a continuously compounded rate of interest r, i.e., the force of interest is r.
- There are no transaction costs, and any assets can be bought and sold (including sold short) in arbitrary quantities.
- We typically assume that the option holder will exercise their option(s) rationally.
- No arbitrage opportunities exist.

A Simple Replicating Portfolio Example

A stock has a current price of \$100. In one period, its price will be \$90 or \$105. We're also given that $r = 0.03$.

We want to use a replicating portfolio to replicate the payoff of a put option (on the above stock) maturing in one period with a strike price of $K = 100$, using the stock and the risk-free asset.

1. What are the possible payoffs of the put option? [0, 10]
2. What is the replicating portfolio for this put option? [-0.6667 stock, 67.93 bond]
3. What is the current (at time 0) price (cost) of the replicating portfolio? [1.26]
4. What is the current (at time 0) price of the put option? [1.26]

Extending the Example

We'll now assume a timeframe of two (independent) periods; the stock price can increase 5% or decrease 10% in each period. The put option still has a strike price of $K = 100$, but now matures at time 2.

We work backwards from time 2 to construct the time 1 replicating portfolio, given the stock price at time 1.

1. What are the possible stock prices and hence payoffs of the put option? [110.25, 94.50, 81; 0, 5.50, 19]
2. What is the price of the put option at time 0? [1.48]

Note that the rebalancing strategy is **self-financing**.

EPV Approach to Option Pricing

We can also use our general EPV formula to calculate the option price, by discounting the option payoff amounts under each path.

The "probabilities" used in the EPV formula for the various payoffs are so-called ***Q*-measure** or **risk-neutral** probabilities.[2]

If d and u represent the possible one-period stock movement factors (e.g., $d = 0.9$, and $u = 1.05$ for this example), then the Q-measure probability of the downward movement occurring in each time period is

$$q = \frac{u - e^r}{u - d}$$

[2]By contrast, the actual probabilities of up and down stock movements are called ***P*-measure** or **real-world** probabilities.

Under the **Black-Scholes-Merton Model**, the log stock returns have Normal distributions under both the P-measure and the Q-measure, though the parameters are different:

$$\ln\left(\frac{S_{t+\tau}}{S_t}\right) \overset{P}{\sim} N\left(\mu\tau, \sigma^2\tau\right)$$

$$\ln\left(\frac{S_{t+\tau}}{S_t}\right) \overset{Q}{\sim} N\left((r-\sigma^2/2)\tau, \sigma^2\tau\right)$$

In general, for an option with payoff $h(S_T)$ at maturity T, the value of the option at time t is

$$v(t) = E_t^Q\left[e^{-r(T-t)} \cdot h(S_T)\right]$$

Put and Call Options

For a call option with strike price K maturing at time T, the price of a call option at time t is

$$c(t) = S_t\Phi(d_1(t)) - Ke^{-r(T-t)}\Phi(d_2(t))$$

and the price of the corresponding put option is

$$p(t) = Ke^{-r(T-t)}\Phi(-d_2(t)) - S_t\Phi(-d_1(t))$$

$$\text{where } d_1(t) = \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$\text{and } d_2(t) = d_1(t) - \sigma\sqrt{T-t}$$

Put-Call Parity

For a put option and a call option with common strike price K maturing at time T, the following relationship is known as the **put-call parity**:

$$c(t) + Ke^{-r(T-t)} = p(t) + S_t$$

This relationship can be derived under the Black-Scholes-Merton framework, but actually holds *regardless of the underlying stock price model*.

Put and Call Options — Replicating Portfolios

For the call option, the replicating portfolio is comprised of:

- A long holding of $\Phi(d_1(t))$ units of the stock, with a value of $S_t\Phi(d_1(t))$
- A short holding of $\Phi(d_2(t))$ zero-coupon bonds with a face value of K, maturing at time T, with a value of $-Ke^{-r(T-t)}\Phi(d_2(t))$

For the put option, the replicating portfolio is comprised of:

- A long holding of $\Phi(-d_2(t))$ zero-coupon bonds with a face value of K, maturing at time T, with a value of $Ke^{-r(T-t)}\Phi(-d_2(t))$
- A short holding of $\Phi(-d_1(t))$ units of the stock, with a value of $-S_t\Phi(-d_1(t))$

Delta and the "Greeks"

The value of an option is based on several different variables: $S_t, K, r, \sigma,$ and $T - t$. All of these except for the strike price are subject to change over the life of the option. Hence, it can be important to know how the price of the option changes as each of these underlying variables change.

The derivatives of the option price with respect to these variables are known as the "Greeks".

The most important of the Greeks is **delta**, which is the derivative of the option price with respective to the underlying stock price.

Delta for Put and Calls

For a European call option: delta $= \dfrac{dc(t)}{dS_t} = \Phi(d_1(t))$

For a European put option: delta $= \dfrac{dp(t)}{dS_t} = -\Phi(-d_1(t))$

Note that — in both cases here, and more generally — delta gives the number of shares of the stock contained in the replicating portfolio.

Thus, the replicating portfolio will consist of (delta $\cdot S_t$) in stock, and ($v(t) -$ delta $\cdot S_t$) in bonds, where $v(t)$ is the price of the option (and also the price of the replicating portfolio).

Other Greeks

Some of the other Greeks commonly calculated (and managed) are:

$$\text{gamma} = \frac{d^2 v(t)}{dS_t^2} = \frac{d \text{ delta}}{dS_t}$$

$$\text{rho} = \frac{dv(t)}{dr}$$

$$\text{tau} = \frac{dv(t)}{d(T-t)} \text{ or } -\frac{dv(t)}{d(T-t)}$$

$$\text{vega} = \frac{dv(t)}{d\sigma}$$

Rebalancing the Replicating Portfolio – Continuous Time

With prices assumed to move continuously, we would need to continuously rebalance our replicating portfolio for it to perfectly hedge the option.

If we were able to do this, the strategy would again be self-financing, but in reality, continuous rebalancing isn't possible.

So we'd likely rebalance at regular discrete intervals; each rebalance could incur a positive or negative cash flow:

- The more frequent we rebalance, the closer our replicating portfolio will track to the option (and the magnitude of the cash flows generated by the rebalance will be smaller), but in practice, each rebalance incurs transaction costs.

VA Guarantees as Put Options

If a VA product has a guarantee like a GMMB, we can view it similarly to a put option:

- The fund plays the role of the stock price, and the guarantee amount is the option strike price.
- The guarantee (option) will be in the money if the fund matures at a level below the guarantee amount. In this instance, the guarantee (option) payoff is the difference between the guarantee and the fund value.
- If the fund exceeds the guarantee amount, the guarantee (option) expires out of the money, and no payment is required.

VA Guarantees as Put Options

But of course, in reality it's not quite that simple, as VA guarantees have additional features that we have to deal with:

- Unlike a regular option, the maturity for a VA guarantee may not be fixed, due to:
  - Deaths under a GMDB guarantee
  - Surrenders, which may nullify a GMxB guarantee
- Also complicating matters is the withdrawal of management charges (fees) from the policyholder's account, impacting the price process of the underlying asset.

In the case of some of the simpler guarantees (like GMMB and GMDB), we can adapt the B-S-M methodology with only relatively minor changes, so that we can do pricing and reserving/hedging.

We'll first look at a simple GMMB that guarantees a maturity benefit of 100% of the single premium P at time n.

We assume some mortality model for the policyholder (x), independent of the asset assumptions. We also assume no surrenders, so that the probability of the policy making it to maturity is ${}_np_x$.

Asset Notation and Assumptions

We carry forward all of the same assumptions as before (no arbitrage, risk-free rate r, etc.)

We make the same assumption of a lognormal price process for the stock, S_t and set $S_0 = 1$. And we let the amount in the fund at time t be denoted by F_t.

We let e denote the initial percent of premium charge, and m denote the percent of fund charge at the start of each subsequent year. Then we have:

$$F_n = P(1-e)(1-m)^{n-1}S_n$$

GMMB Pricing

As we saw previously, under the B-S-M model, we can think about the price of an option with payoff $h(S_T)$ at time T using an EPV-style approach:

$$E_t^Q\left[e^{-r(T-t)} \cdot h(S_T)\right]$$

For a put option, this is

$$E_t^Q\left[e^{-r(T-t)} \cdot (K - S_T)^+\right]$$

Applying this to our GMMB, the cost at issue is

$$\pi(0) = {}_n p_x E_0^Q\left[e^{-rn}(P - F_n)^+\right]$$

GMMB Pricing (Continued)

With some work, we can get an expression in the form of a put option on the underlying stock:

$$\pi(0) = P_n p_x \xi E_0^Q \left[e^{-rn} (\xi^{-1} - S_n)^+ \right]$$

where $\xi = (1-e)(1-m)^{n-1}$ is a constant whose value will be known at issue. And so, finally:

$$\pi(0) = P_n p_x \left(e^{-rn} \Phi(-d_2(0)) - \xi \Phi(-d_1(0)) \right)$$

$$d_1(0) = \frac{\log(\xi) + (r + \sigma^2/2)n}{\sigma\sqrt{n}} \quad \text{and} \quad d_2(0) = d_1(0) - \sigma\sqrt{n}$$

GMMB Pricing (Continued)

More generally:

$$\pi(0) = {}_n p_x \left(K e^{-rn} \Phi(-d_2(0)) - S_0 \xi \Phi(-d_1(0)) \right)$$

$$d_1(0) = \frac{\log\left(\frac{S_0 \xi}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) n}{\sigma \sqrt{n}} \quad \text{and} \quad d_2(0) = d_1(0) - \sigma \sqrt{n}$$

Example 17.1

An insurer issues a 10-year VA with a single premium of $P = 10,000$ to (60). An expense charge of 3% of the premium is deducted from the premium. Also, an annual management charge of 0.5% is deducted at the start of each renewal year.

The only guarantee is a GMMB, which is 100% of the premium.

Assume that the underlying stock follows a lognormal return process with volatility $\sigma = 0.25$; also assume that $r = 0.05$. We assume no lapses and that mortality is given by the SULT.

Calculate the cost at issue of the GMMB, as a percent of the single premium. [0.1002P]

Similar to above, we can use B-S-M to get the cost of the guarantee at time t, which we can use as the time t reserve:

$$\pi(t) = P \, {}_{n-t}p_{x+t} \left(e^{-r(n-t)} \Phi(-d_2(t)) - \xi S_t \Phi(-d_1(t)) \right)$$

$$d_1(t) = \frac{\ln(\xi S_t) + (r + \sigma^2/2)(n-t)}{\sigma\sqrt{n-t}} \quad \text{and} \quad d_2(t) = d_1(t) - \sigma\sqrt{n-t}$$

Example 17.2

Now assume that the policy is still in force six years after issue.

Again assuming no lapses, calculate the reserve for the GMMB at this time, if the value of the underlying stock has:

(a) increased by 45% [358.92]

(b) increased by 5% [905.39]

GMDB Pricing

We can approach GMDB guarantees similarly to GMMB guarantees; the biggest difference is that with a GMDB, we don't know beforehand when the payout will be.

If we knew the payout of $h(t)$ was occurring at time t, the time 0 value would be

$$v(0,t) = E_0^Q\left[e^{-rt}h(t)\right]$$

Thus, its price, considering the uncertainty of time of death, is

$$\pi(0) = \int_0^n v(0,t)\, {}_t p_x \, \mu_{x+t}\, dt$$

If the payoff is at the end of the month of death, we have

$$\pi(0) = \sum_{j=1}^{12n} v(0, j/12)\, {}_{\frac{j-1}{12}|\frac{1}{12}} q_x$$

Example 17.3

An insurer issues a 5-year VA with a single premium of $P = 10,000$ to (60). A management charge of 0.25% is deducted at the start of each month.

The only guarantee is a GMDB, which is set to the accumulated amount of the premium with interest at $r = 0.05$, at the end of the month of death.

Assume that the underlying stock follows a lognormal return process with volatility $\sigma = 0.25$. We assume no lapses and that mortality is given by the Makeham model with $A = 0.0001, B = 0.00035$, and $c = 1.075$.

Calculate the cost at issue of the GMDB.

Similar to above, we can use B-S-M to get the cost of the guarantee at time t, which we can use as the time t reserve.

If the payoff is known to be at time s, then the value at time t is:

$$v(t,s) = E_t^Q\left[e^{-r(s-t)}h(s)\right]$$

Considering the uncertainty of time of death, the value at time t is:

$$\pi(t) = \int_t^n v(t,s)\,{}_{s-t}p_{x+t}\,\mu_{x+s}\,ds$$

Or in the monthly case,

$$\pi(t) = \sum_{j=1}^{12(n-t)} v(t, t+j/12)\,{}_{\frac{j-1}{12}|\frac{1}{12}}q_{x+t}$$

Example 17.4

Now assume that the policy is still in force three and a half years after issue.

Again assuming no lapses, calculate the reserve for the GMDB at this time, if the value of the underlying stock has:

(i) increased by 50% since issue, so that $S_{3.5} = 1.5$

(ii) has remained at its initial value, so that $S_{3.5} = 1$

Variable Annuity Pricing

The pricing of variable annuities largely comes down to determining a reasonable charge for the product's guarantees.

In the U.S., the most common fee structure is to have **M&E (Mortality and Expense) Fee**, deducted monthly or annually, and expressed as a percentage of the fund value.

There could also be an annual **policy fee** that's a flat amount; sometimes it's waived for accounts exceeding some specified minimum balance.

Monte Carlo (MC) simulation methodologies for pricing options and "equity-linked contracts" started in the 1970s (Boyle, 1977; Boyle and Schwartz, 1977).

Profit Testing of GMxB

For a profit test of a GMxB, our inflows will be the expense charges.

Outflows will include our actual expenses, the payoffs of the guarantees, and the costs associated with hedging the guarantee.

- We may choose to hedge internally, or by purchasing options on the open market.

These inflows and outflows comprise the profit vector, from which we can calculate the profit signature and profit measures.

We could do the profit test under one or more deterministic stock market scenarios, but we should consider performing stochastic Monte Carlo simulations to get a distribution of results.

We can test different levels of charges and assess their adequacy.

Variable Annuity Hedging Issues

How often to rebalance?

How to account for policyholder behavior?

Basis risk refers to the difference between the index being hedged and the index with the actual exposure.

Gap risk occurs when the index price changes significantly before the hedge can be rebalanced.

The selection of funds offered to the policyholder in the VA product may be influenced by hedging considerations.

As always, product design is an important tool in helping to manage the risks.

Hedging may also help smooth profits and losses.

Sources

- “Variable Annuity Guarantees.” Mary Hardy (2022)
- Sun et al. “How effective is variable annuity hedging?” Milliman (2016)
- Bergantino, M. “Variable Annuities Risk Management.” Risk and Investment Conference (2012)
- Boyle, Phelim P. "Options: A Monte Carlo approach." Journal of Financial Economics 4.3 (1977): 323-338.
- Boyle, Phelim P., and Eduardo S. Schwartz. “Equilibrium Prices of Guarantees under Equity-Linked Contracts.” The Journal of Risk and Insurance 44 (1977)