Stat 444 - Hartman **Name:** ______________________

Practice Midterm Exam

This exam contains 11 pages (including this cover page) and 6 problems. Check to see if any pages are missing.

You may only use SOA-approved calculators and a pencil or pen on this exam.

You are required to show your work on each problem on this exam.

Grade calculation errors: If I made an arithmetic mistake (I miscounted your total points) please come and see me and I will fix it.

Regrade requests: I make every effort to grade your test (and those of your classmates) fairly. If you feel I graded a portion of your test too harshly, please write an explanation on the back of the test and turn it into me by Thursday, March 7th in class. Please note that to maintain fairness your entire test will be regraded, potentially resulting in a lower overall grade.

Problem	Points	Score
1	20	
2	16	
3	27	
4	18	
5	20	
6	9	
Total:	110	

1. Consider a four-state discrete-time Markov model with states Single, Married, Divorced, and Widowed. The annual probability transition matrix is below:

$$\begin{array}{c} \\ S \\ M \\ D \\ W \end{array} \begin{array}{c} \begin{array}{cccc} S & M & D & W \end{array} \\ \begin{pmatrix} 0.75 & c & 0 & 0 \\ 0 & 0.80 & 0.15 & 0.05 \\ 0 & 0.15 & 0.85 & 0 \\ 0 & 0.10 & 0 & 0.90 \end{pmatrix} \end{array}$$

a. [2 pts] Calculate c.

b. [6 pts] Calculate the probability that someone currently Married will be Divorced two years from now.

A two-year term insurance policy is issued only to a Married person and makes a payment of \$10,000 upon the insured becoming Widowed. (The policy will make a maximum of one payment.) Assume that $i = 0.07$ and that the only expenses are a \$500 issue expense (payable at issue) and a \$100 claim expense (payable at the same time as the benefit payment).

c. [6 pts] Calculate the gross single premium for this insurance.

d. [6 pts] Calculate the gross premium reserve at time 1 if the insured is Married at that time.

Answer:

(a) $c = Pr(M_1|S_0) = 1 - [Pr(S_1|S_0) + Pr(D_1|S_0) + Pr(W_1|S_0)]$
$c = 1 - (0.75 + 0 + 0)$
$c = 0.25$

(b) Pr("Divorced in 2 years" — "Married now") =
= $\Pr(D_1|M_0) * \Pr(D_2|D_1) + \Pr(M_1|M_0) * \Pr(D_2|M_1)$ =
= .15(.85) + .8(.15) = .1275 + .12 = 0.2475

(c) P = 10100 $A^{MW}_{x^1:\overline{2}|}$ + 500
P = $10100[v * \Pr(W_1|M_0) + v^2 * \Pr(M_1|M_0) * \Pr(W_2|M_1)]$
P = $10100(1.07^{-1} * .05 + 1.07^{-2} * .8 * .05) + 500$
P = 10100(.08167) + 500
P = 1324.83

(d) ${}_tV^{(M)} = 10100\ A^{MW}_{x+1^1:\overline{1}|}$
${}_tV^{(M)} = 10100(v * \Pr(W_2|M_1))$
${}_tV^{(M)} = 10100(1.07^{-1} * .05)$
${}_tV^{(M)} = 471.96$

2. Consider a \$100,000 fully-discrete, second-to-die, whole life insurance policy issued to (75) and (75). (That is, the policy pays a death benefit at the end of the year of the second death.) Assume that mortality for these individuals is given by the SULT, they have independent future lifetimes, and that $i = 5\%$.

   Premiums are payable while both insureds are alive, but for a maximum of 10 years.

   Expenses for this policy are:

   - \$500 at issue
   - 3% of gross premiums

   a. Calculate the following probabilities:
      i. [2 pts] ${}_2q_{75}$
      ii. [2 pts] ${}_2q_{75:75}$
      iii. [2 pts] ${}_2q_{\overline{75:75}}$

   b. [7 pts] Calculate the gross annual premium for the policy.

   c. [3 pts] Calculate the gross premium reserve at duration 10 for this policy, i.e., ${}_{10}V^g$, assuming that only one insured is alive at this time.

Answer:

(a) i. ${}_2q_{75} = 1 - \frac{l_{77}}{l_{75}}$
${}_2q_{75} = 1 - \frac{81904.3}{85203.5}$
${}_2q_{75} = 0.03872$

ii. ${}_2q_{75:75} = 1 - ({}_2p_{75} * {}_2p_{75})$
${}_2q_{75:75} = 1 - (\frac{l_{77}}{l_{75}} * \frac{l_{77}}{l_{75}})$
${}_2q_{75:75} = 1 - (\frac{81904.3}{85203.5} * \frac{81904.3}{85203.5})$
${}_2q_{75:75} = 0.07594$

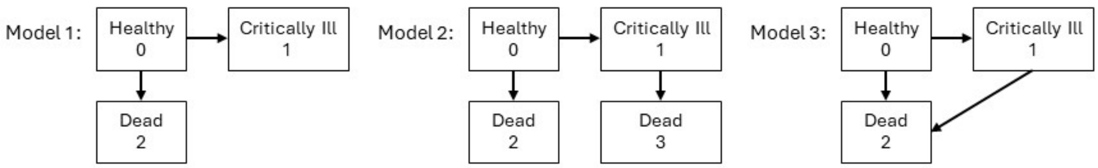
iii. ${}_2q_{\overline{75:75}} = {}_2q_{75} * {}_2q_{75}$
${}_2q_{\overline{75:75}} = .03872 * .03872$ (from part i)
${}_2q_{\overline{75:75}} = 0.0015$

(b) First, calculate some values that we'll need:
$A_{\overline{75:75}} = A_{75} + A_{75} - A_{75:75}$
$A_{\overline{75:75}} = .50868 + .50868 - .60912$
$A_{\overline{75:75}} = 0.40828$
$\ddot{a}_{75:75\overline{10|}} = 6.6563$
P = $(500 + 100{,}000 * A_{\overline{75:75}}) / (.97 * \ddot{a}_{75:75\overline{10|}})$
P = (500 + (100,000 * .40824)) / (.97 * 6.6563)
P = 6400.26

(c) ${}_{10}V^{(g)} = 100000 * A_{85} - 0$
${}_{10}V^{(g)} = 100000(.67622)$
${}_{10}V^{(g)} = 67622$

3. An insurance company offers the following range of options for critical illness (CI) insurance riders attached to a life insurance policy. In each case the critical illness benefit would be paid immediately upon diagnosis of a covered critical illness.

- Option A: Partially accelerated death benefit
- Option B: Fully accelerated death benefit
- Option C: Additional lump sum benefit on CI diagnosis



a. [3 pts] For each of the options above, state with reasons which of the models above could be used to calculate premiums and reserves.

A special 10-year term insurance policy issued to a Healthy life age 50 pays 100,000 immediately upon death of the policyholder, and an additional lump sum benefit of 50,000 immediately upon diagnosis of critical illness. Premiums are payable annually while the policyholder is Healthy. The insurer uses Model 3 for analyzing this policy.

b. [2 pts] Write down an integral expression for the probability that a new policyholder dies before the policy expires.

c. [2 pts] Write down an integral expression for the expected present value of the death benefit.

The insurer values the contract payments using force of interest of $\delta = 0.05$. You are given the following table of actuarial functions under Model 3, at $\delta = 0.05$.

x	${}_{10}p_x^{00}$	${}_{10}p_x^{01}$	μ_x^{01}, μ_x^{02}	$\bar{a}_x^{00}$	$\bar{A}_x^{01}$	$\bar{A}_x^{02}$	$\bar{A}_x^{12}$
50	0.90880	0.04152	0.00562	13.1491	0.21846	0.21878	0.30227
60	0.75019	0.13565	0.01542	10.0781	0.33692	0.33714	0.41861

d. [5 pts] Calculate $\ddot{a}_{50:\overline{10}|}^{00}$ using Woolhouse's 3-term formula.

e. [8 pts] Calculate the annual net premium.

f. [4 pts] You are given the following additional information for a policy in force at time 5:

t	${}_tV^{(0)}$	${}_tV^{(1)}$	μ_{50+t}^{01}	μ_{50+t}^{02}
5	702.6	5322.0	0.004265	0.004835
6	714.3	4562.4	0.004845	0.005234

Calculate $\frac{d}{dt}{}_tV^{(0)}$ at time 5, immediately after the premium payment.

g. [3 pts] Sketch the graph of ${}_tV^{(0)}$ as a function of t for $5 \le t \le 6$. You should mark key numerical values on each axis.

Answer:

a. For Option A, Model 2 could be used because it allows for a payment upon CI diagnosis, and then two different payment amounts on death, depending on whether the CI benefit had previously been paid. Model 1 doesn't work for this option because if the CI benefit is paid, the model doesn't allow for a subsequent DB payment; Model 3 doesn't work because it does not distinguish between deaths after CI or deaths without a CI diagnosis, which matters for this option.

For Option B, Model 1 or Model 2 or Model 3 could work.

For Option C, Either Model 2 or Model 3 could work. Model 1 would not work for this option because it only allows for a single payment, and two payments are necessary in the event that the insured receives a CI diagnosis prior to death.

b.

$$\int_0^{10} {}_tp_{50}^{00}\mu_{50+t}^{02} + {}_tp_{50}^{01}\mu_{50+t}^{12}dt$$

c.

$$100,000v^t \int_0^{10} {}_tp_{50}^{00}\mu_{50+t}^{02} + {}_tp_{50}^{01}\mu_{50+t}^{12}dt$$

d.

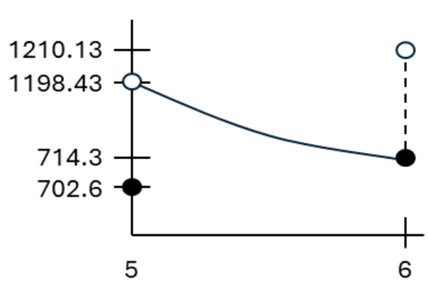
$$\begin{aligned}
\ddot{a}_{50}^{00} &\approx \bar{a}_{50}^{00} + 1/2 + \frac{\mu_{50}^{01} + \mu_{50}^{02} + \delta}{12} = 13.65374 \\
\ddot{a}_{60}^{00} &\approx \bar{a}_{60}^{00} + 1/2 + \frac{\mu_{50}^{01} + \mu_{60}^{02} + \delta}{12} = 10.58355 \\
\ddot{a}_{50:\overline{10}|}^{00} &= \ddot{a}_{50}^{00} - e^{-10\delta}{}_{10}p_{50}^{00}\ddot{a}_{60}^{00} \approx 7.8199
\end{aligned}$$

e.

$$\begin{aligned}
\text{EPV death benefit} &= 100,000(\bar{A}_{50}^{02} - e^{-10\delta}{}_{10}p_{50}^{00}\bar{A}_{60}^{02} - e^{-10\delta}{}_{10}p_{50}^{01}\bar{A}_{60}^{12}) \\
&= 100,000(0.21878 - e^{-0.5}(0.9088)(0.33174) - e^{-0.5}(0.04512)(0.41861)) = 2240.14 \\
\text{EPV CI benefit} &= 50,000(\bar{A}_{50}^{01} - e^{-10\delta}{}_{10}p_{50}^{00}\bar{A}_{60}^{01}) \\
&= 50,000(0.21846 - e^{-0.5}(0.9088)(0.33692)) = 1637.23 \\
\text{EPV Premiums} &= 7.8199P \to P = \frac{2240.16 + 1637.23}{7.8199} = 495.83
\end{aligned}$$

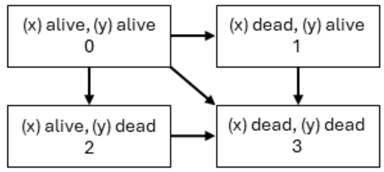
f.

$$\begin{aligned}
\left.\frac{d}{dt}{}_tV^{(0)}\right|_{t=5^+} &= {}_{5^+}V^{(0)}\delta - \mu_{55}^{01}(50,000 + {}_5V^{(1)} - {}_{5^+}V^{(0)}) - \mu_{55}^{02}(100,000 - {}_{5^+}V^{(0)}) \\
{}_tV^{(0)} &= 702.6 + 495.8 = 1198.4 \\
\left.\frac{d}{dt}{}_tV^{(0)}\right|_{t=5^+} &= 1198.4(0.05) - 0.004265(50,000 + 5322 - 1198.4) - 0.004835(100,000 - 1198.4) \\
&= -648.6
\end{aligned}$$



g.

4. Consider the following multi-state model for modeling joint life insurance policies issued to a couple (x) and (y):



   a. [2 pts] State two reasons why a couple may have dependent future lifetimes.
   b. [2 pts] State the conditions for (x) and (y) to have independent future lifetimes, in terms of $\mu^{01}_{x+t:y+t}, \mu^{02}_{x+t:y+t}, \mu^{13}_{y+t}, \mu^{23}_{x+t}$, and $\mu^{03}_{x+t:y+t}$
   c. i. [2 pts] Write down the Kolmogorov forward differential equation, with the associated boundary condition, for ${}_tp^{00}_{x:y}$.
      ii. [2 pts] Use the Kolmogorov forward differential equation to show that ${}_tp^{00}_{x:y} = \exp\left(-\int_0^t \mu^{01}_{x+g:y+g} + \mu^{02}_{x+g:y+g} + \mu^{03}_{x+g:y+g} dg\right)$

A couple, (x) who is age 40 and (y) who is age 50, buys a fully discrete last survivor whole life insurance with a sum insured of 100,000. You are given:

- The couple's future lifetimes are independently distributed.
- Premiums are payable while at least one life is alive for a maximum of 10 years.
- Mortality follows the Standard Ultimate Life Table (SULT).
- $i = 0.05$

   d. [4 pts] Calculate the annual net premium.
   e. [2 pts] Calculate the policy value at time 10, if (y) is alive but (x) is dead.
   f. [2 pts] Calculate the policy value at time 10, if both (x) and (y) are alive.
   g. [2 pts] The insurer decides to incorporate a common shock risk in the joint life model, although each life's marginal individual mortality still follows the SULT. State with reasons whether the policy values in (e) and (f) would increase, decrease, or stay the same as a result of the common shock risk.

Answer:

a.
- A couple may have common lifestyles leading to similar mortality (eg both smokers)
- The broken heart syndrome increases mortality of the surviving partner in the period following their spouse's death
- The couple may be exposed to common shock risk, i.e., the risk that they die at the same time, e.g., in an accident.

b. Two conditions:
- $\mu^{03}_{x+t:y+t} = 0$ - common shock
- $\mu^{02}_{x+t:y+t} = \mu^{13}_{y+t}$ and $\mu^{01}_{x+t:y+t} = \mu^{23}_{x+t}$ - common lifestyle/broken heart

c. i. $\frac{d}{dt}{}_tp^{00}_{xy} = -{}_tP^{00}_{xy}(\mu^{01}_{x+t:y+t} + \mu^{02}_{x+t:y+t} + \mu^{03}_{x+t:y+t})$, Boundary Cond. ${}_0P^{00}_{xy} = 1$

ii.

$$\begin{aligned}
\frac{d}{dt}{}_tp_{xy}^{00} &= -{}_tP_{xy}^{00}(\mu_{x+t:y+t}^{01} + \mu_{x+t:y+t}^{02} + \mu_{x+t:y+t}^{03}) \\
\frac{1}{{}_tp_{xy}^{00}}\frac{d}{dt}{}_tp_{xy}^{00} &= -(\mu_{x+t:y+t}^{01} + \mu_{x+t:y+t}^{02} + \mu_{x+t:y+t}^{03}) \\
\frac{d}{dt}\log {}_tp_{xy}^{00} &= -(\mu_{x+t:y+t}^{01} + \mu_{x+t:y+t}^{02} + \mu_{x+t:y+t}^{03}) \\
\int_0^n \frac{d}{dt}\log {}_tp_{xy}^{00}dt &= -\int_0^n(\mu_{x+t:y+t}^{01} + \mu_{x+t:y+t}^{02} + \mu_{x+t:y+t}^{03})dt \\
\log {}_np_{xy}^{00} - \log {}_0p_{xy}^{00} &= -\int_0^n(\mu_{x+t:y+t}^{01} + \mu_{x+t:y+t}^{02} + \mu_{x+t:y+t}^{03})dt \\
\log {}_np_{xy}^{00} - \log 1 &= -\int_0^n(\mu_{x+t:y+t}^{01} + \mu_{x+t:y+t}^{02} + \mu_{x+t:y+t}^{03})dt \\
\log {}_np_{xy}^{00} &= -\int_0^n(\mu_{x+t:y+t}^{01} + \mu_{x+t:y+t}^{02} + \mu_{x+t:y+t}^{03})dt \\
{}_np_{xy}^{00} &= \exp\left\{-\int_0^n(\mu_{x+t:y+t}^{01} + \mu_{x+t:y+t}^{02} + \mu_{x+t:y+t}^{03})dt\right\}
\end{aligned}$$

d.

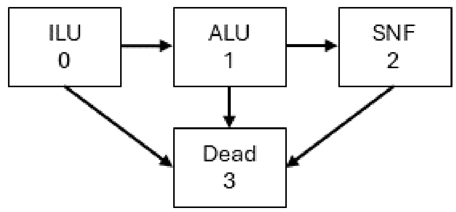
$$\begin{aligned}
P\ddot{a}_{\overline{40:50}:\overline{10|}} &= 100,000(A_{\overline{40:50}}) \\
\ddot{a}_{\overline{40:50}:\overline{10|}} &= \ddot{a}_{40:\overline{10|}} + \ddot{a}_{50:\overline{10|}} - \ddot{a}_{40:50:\overline{10|}} = 8.1076 \\
A_{\overline{40:50}} &= A_{40} + A_{50} - A_{40:50} = 0.098737 \\
P &= 1217.83
\end{aligned}$$

e. ${}_{10}V^{(1)} = 100,000A_{60} = 29,028$

f. ${}_{10}V^{(0)} = 100,000A_{\overline{50:60}} = 15,911.10$

g. The policy value in (e) would stay the same. Once (x) has died, the policy value only depends on the marginal mortality of (y), which is unchanged. The policy value in (f) would increase. The time to the first death is the same as under the independent model, but the time to the second death is less than or equal to the independent model, because of the possibility of simultaneous deaths.

5. A Continuing Care Retirement Community (CCRC) offers new residents Modified Life Care contracts. The CCRC contains three different types of accommodation: Independent Living Units (ILU), Assisted Living Units (ALU), and Specialized Nursing Facilities (SNF). The following Markov model is used to determine fees.



You are given:

- The CCRC incurs the following monthly costs per resident, payable continuously, by type of accommodation.
  ILU: 3,000
  ALU: 7,000
  SNF: 15,000
- Each resident pays a lump sum fee immediately on entry and pays residence fees continuously while in the CCRC.
- Residence fees are a fixed proportion, γ, of the monthly costs.
- The value of γ is determined such that the EPV of future costs is equal to the EPV of all fees, including the entry fee.
- $i = 0.05$
- The following actuarial functions, evaluated at 5%.

X	$\bar{a}_x^{00}$	$\bar{a}_x^{01}$	$\bar{a}_x^{02}$	$\bar{a}_x^{11}$	$\bar{a}_x^{12}$	$\bar{a}_x^{22}$
65	11.4106	1.3570	0.3745	11.8352	0.7979	10.6905
70	9.5210	1.7037	0.4942	10.1754	0.9960	9.1961

An individual enters the CCRC, in an ILU, at age 65. Their entry fee is 150,000.

a. [2 pts] Calculate the expected present value of future costs.

b. [4 pts] Calculate γ.

Let ${}_tV_x^{(j)}$ denote the EPV at time t of the CCRC's future costs minus future income, for an individual who entered the CCRC at age x, and who is now in state j at age $x + t$.

c. [4 pts] Calculate ${}_5V_{65}^{(1)}$.

You are also given:

- ${}_5V_{65}^{(2)} = 419,280$
- $\mu_{70}^{12} = 0.00156$ and $\mu_{70}^{13} = 0.01194$

d. i. [2 pts] Write down Thiele's differential equation for ${}_tV_{65}^{(1)}$.

ii. [2 pts] Calculate $\frac{d}{dt}{}_tV_{65}^{(1)}$.

iii. [4 pts] Using Euler's forward method, with a step size of h = 0.25, calculate ${}_{5.25}V_{65}^{(1)}$.

e. [2 pts] The CCRC introduces a Full Life Care contract option, under which residence fees are level throughout their residency, regardless of the accommodation type. Entry fees are the same for both contract options. State with reasons whether an individual would pay a higher residence fee while in ILU under the Full Life Care contract or the Modified Life Care contract.

Answer:

a. The expected present value of future costs for (65) is

$$EPV = 12(3000\bar{a}_{65}^{00} + 7000\bar{a}_{65}^{01} + 15000\bar{a}_{65}^{02}) = 592,180$$

b. The parameter γ satisfies

$$EPV = 12\gamma(3000\bar{a}_{65}^{00}+7000\bar{a}_{65}^{01}+15000\bar{a}_{65}^{02})+150,000 \rightarrow \gamma = \frac{592,179.6 - 150,000}{592,179.6} = 0.7467$$

c.

$$_5V_{65}^{(1)} = 12(7000(1-\gamma)\bar{a}_{70}^{11} + 15000(1-\gamma)\bar{a}_{70}^{12}) = 261,917$$

d. i. Thiele's equation for $_tV_{65}^{(1)}$ is

$$\frac{d}{dt}{}_tV_{65}^{(1)} = \delta\ {}_tV_{65}^{(1)} - 12(1-\gamma)7000 - \mu_{65+t}^{12}({}_tV_{65}^{(2)} - {}_tV_{65}^{(1)}) - \mu_{65+t}^{13}(0 - {}_tV_{65}^{(1)})$$

ii.

$$\left.\frac{d}{dt}{}_tV_{65}^{(1)}\right|_{t=5} = \delta \times 261,917 - 12(1-0.7467)7000 - 0.00156(419,280 - 261,917)$$
$$-0.011940(0 - 261,917.2) = -5616.5$$

iii. Using Euler's forward method,

$$_{5+h}V_{65}^{(1)} \approx {}_5V_{65}^{(1)} + h\left.\frac{d}{dt}{}_tV_{65}^{(1)}\right|_{t=5} = 261,917.2 + 0.25(-5,616.54) = 260,513.1$$

e. The ILU residence fee is higher under the Full Life Care contract. Both the EPV of costs and the entry fee are the same for an individual, so the EPV of total residence fees under the Full Life Care and modified life care are the same, but the modified life care fees step up on transition out of the ILU. The FLC effectively spreads that increase over the whole residency period, resulting in higher fees in the ILU.

6. You are given the following excerpt from a triple decrement table:

x	ℓ_x	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
60	1000	$d_{60}^{(1)}$	60	45
⋮	⋮	⋮	⋮	⋮

You are also given the following information about the decrements:

- $\mu_{60+t}^{(1)} = 1.2t$ for $0 \le t \le 1$
- Decrement 2 happens exactly halfway through the year.
- Decrement 3 happens at the end of the year.

a. [4 pts] Calculate $q_{60}^{'(2)}$

b. [2 pts] Calculate $d_{60}^{(1)}$.

Now suppose instead that Decrement 2 occurs at the start of the year, and that each $q_{60}^{'(i)}$ remains unchanged.

c. [3 pts] State with reasons the effect (increase, decrease, no change, cannot be determined) that this change would have on the following probabilities: $q_{60}^{(1)}$, $q_{60}^{(2)}$, and $q_{60}^{(3)}$.

Answer:

a. Let $\ell_{x+t^-}^{(\tau)}$ denote the value immediately before exits at exact age $x+t$, and $\ell_{x+t^+}^{(\tau)}$ denote the value immediately after. At $t = 0.5$ just before decrement 2 exits, we have

$$q_{60}^{'(2)} = \frac{d^{(2)}}{\ell_{60.5^-}^{(\tau)}}$$

$$\ell_{60.5^-}^{(\tau)} = 1000 \exp\left(-\int_0^{0.5} 1.2t dt\right) = (1000)(0.086071 = 860.71)$$

$$q_{60}^{'(2)} = \frac{60}{860.71} = 0.0697$$

b.

$$\ell_{60.5^+}^{(\tau)} = 800.71, \ell_{61^-}^{(\tau)} = 800.71 \exp\left(-\int_{0.5}^{1} 1.2t dt\right) = 800.71(0.63763) = 510.6$$

$$\ell_{61^+}^{(\tau)} = 510.6 - 45 = 465.6$$

$$d_{60}^{(1)} = \ell_{60}^{(\tau)} - d_{60}^{(2)} - d_{60}^{(3)} - \ell_{61^+}^{(\tau)} = 1000 - 60 - 45 - 466 = 429$$

c. (i) If decrement 2 occurs at the start of the year, there are fewer lives exposed to force of decrement 1, so $q_{60}^{(1)}$ would be smaller.
(ii) If decrement 2 occurs at the start of the year, there are more lives exposed to force of decrement 2, so $q_{60}^{(2)}$ would be bigger.
(iii) Because all the decrement 3 exits happen at the end of the year, we have

$$q_{60}^{(3)} = \frac{d_{60}^{(3)}}{\ell_{60}^{(\tau)}} \quad \text{and} \quad q_{60}^{'(3)} = \frac{d_{60}^{(3)}}{\ell_{61^-}^{(\tau)}}$$

where $\ell_{61-}^{(\tau)}$ is the expected number of in-force immediately before the decrement 3 exits at the end of the year. Also $\ell_{61-}^{(\tau)} = \ell^{(\tau)} p_{60}^{\prime(1)} p_{60}^{\prime(2)}$ and since the independent rates are unchanged, $\ell_{61-}^{(\tau)}$ is unchanged, which means that $d_{60}^{(3)}$ is unchanged, which means that $q_{60}^{(3)}$ is unchanged.