BRIGHAM YOUNG UNIVERSITY

STAT 446 ADVANCED SHORT-TERM ACTUARIAL MODELING

ASSIGNMENT 4

SEVERITY MODELS

1. An actuary has performed a longitudinal study on a group of patients and finds that the conditional hazard rate function of a patient's future lifetime is

$$h_{X|\Lambda}(x|\lambda) = 3\lambda x^2, \qquad x > 0,\ \lambda > 0,$$

where Λ is gamma distributed with $\alpha = 2$ and $\theta = 0.01$.

   (a) Determine the conditional survival function of the future lifetime for a patient in the group. [$e^{-\lambda x^3}$]

   (b) Calculate the probability that a randomly selected patient in the group survives more than 5 years. [0.1975]

2. A loss follows a spliced distribution. The density function below 500 is the same as the density function of an exponential distribution with mean $\theta = 250$. The density function of the loss beyond 500 is proportional to the density function of a Weibull distribution with parameters $\tau = 2$ and $\theta = 400$.

   (a) Determine the density function of the loss. [proportionality constant $c = 0.6456$]

   (b) Calculate the probability that the loss is no more than 600. [0.9320]

   (c) Determine VaR of the loss at the 95% security level. [639.78]

3. You are given that the random variable X has probability density function

$$f(x) = (1 + 2x)e^{-2x}, \qquad x > 0.$$

   (a) Calculate the mean of X. [0.75]

   (b) Determine the survival function $S(x)$. [$(1 + x)e^{-2x}$]

   (c) Determine the hazard rate function $h(x)$, and classify the tail of X based on $h(x)$. [$h(x) = \frac{1+2x}{1+x}$, increasing: light tail]

   (d) Determine the mean excess loss function $e(d)$, and classify the tail of X based on $e(d)$. [$e(d) = \frac{0.75+0.5d}{1+d}$, decreasing: light tail]

   (e) Using the limiting ratio of the survival functions, compare the relative tail weight of X to a Pareto random variable with parameters $\alpha = 3$ and $\theta = 0.25$. [limit is 0: X has a lighter tail]

4. Consider the Pareto distribution with distribution function

$$F_X(x) = 1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha}.$$

(a) Show that this distribution is in the Fréchet MDA, using normalizing sequences $c_n = \frac{\theta}{\alpha} n^{1/\alpha}$ and $d_n = \theta\left(n^{1/\alpha} - 1\right)$.

(b) Given $\theta = 2000$ and $\alpha = 3.0$,

(i) Compare the exact probability that $M_n \leq m$ with the probability approximated by the GEV distribution, for $n = 25$ and $m = 5000$. [exact 0.5543; GEV 0.5582]

(ii) Repeat (i) for $m = 5000$ and $n = 100$. [exact 0.0944; GEV 0.0971]

(iii) Repeat (i) with $n = 25$ and $m = 10{,}000$. [exact 0.8905; GEV 0.8907]

(iv) Comment on your results.

5. A company is modelling losses from cyberattacks. The following table shows the largest 20 values of a sample of 1,000 observations of the losses, sorted in decreasing order:

196.1	148.2	79.8	35.8	27.1	22.9	21.8	20.9	16.7	15.8
15.5	13.7	13.0	12.5	11.5	10.3	9.7	9.4	8.3	8.0

(a) Estimate the 99% VaR of X. [15.80]

(b) Estimate the 99% Expected Shortfall of X. [58.51]

(c) Assume that the losses are from a distribution which is GPD for $Y = X - d \,|\, X > d$, where $d = 10$, with shape parameter $\xi = 0$, and scale parameter $\beta = 30$.

(i) Show that

$$\Pr(X > x) = \overline{F}_X(d)\left(e^{-(x-d)/\beta}\right)$$

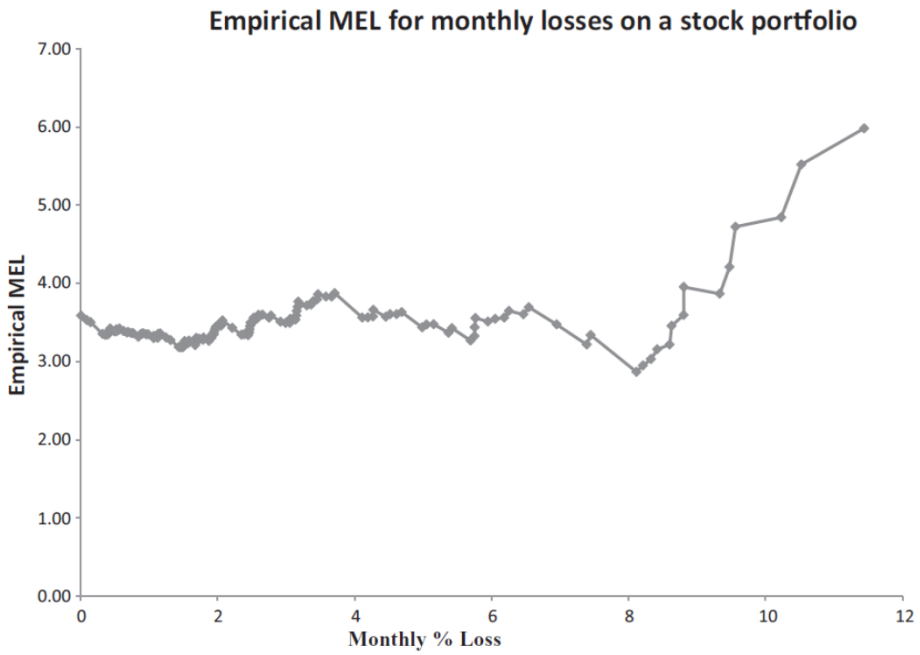
for $x > d$.

(ii) Estimate the 99% VaR and 99% Expected Shortfall using the GPD for the tail probabilities. [24.10; 54.10]

(iii) Comment on the differences between your estimates in (a) and (b), and your estimates using the GPD.

6. An investment firm is interested in fitting an extreme value distribution to portfolio losses. The data set comprises 300 values for the monthly percentage loss in portfolio value.

The analyst has assumed the data is a random sample of independent identically distributed observations. She has calculated the empirical MEL function, which is shown in the figure below.



You are also given the 20 largest values from the data, sorted from high to low:

24.3	18.4	15.6	11.4	10.5	10.2	9.6	9.5	9.3	8.8
8.8	8.6	8.6	8.4	8.3	8.2	8.1	7.4	7.4	6.9

(a) (i) Estimate the 98% Expected Shortfall from the data. [15.07]

(ii) Calculate the empirical MEL function, $\hat{e}(x)$ at $x = 10.0$. [5.07]

(b) The analyst decides to use the GPD with threshold $d = 8$. Explain why this choice appears reasonable.

(c) The MLE estimators for the GPD distribution, with $d = 8$, are $\hat{\xi} = 0.55$, $\hat{\beta} = 1.5$.

(i) Calculate the 98% Expected Shortfall using the GPD distribution. [16.02]

(d) Comment on the difference between the Expected Shortfall estimates from (a)(i) and (c)(i). Which method would you recommend?