Stat 446
Advanced Short-Term Actuarial Modeling

Chapter 1: Policy Modification

Outline

1. Severity: Policy Limits
   - Policy Limits
   - Increased Limit Factor
2. Severity: Policy Deductibles
   - Policy Deductibles
   - Loss Elimination Ratio
3. Severity: Coinsurance, Inflation and All Together
   - Putting Together
4. Coverage Modifications on Frequency

Claim Adjustment

- Insurance policies contain adjustments to determine the amount paid by the insurer on a given loss.
- The most common policy adjustments are listed below:
  - **Policy limit**: for any given loss X, the maximum amount paid or covered loss by the insurer (in the textbook, policy limit u refers to the maximum covered loss).
  - **Policy deductible**: for any given loss, the first d dollars fall on the policyholder (i.e., eliminates a large volume of very small claims).
  - **Coinsurance**: the insurer pays a proportion α of the loss; the remaining $(1 - \alpha)$ is covered by the policyholder (i.e., cost sharing).
- In this chapter, we will investigate the impact of these policy modifications, as well as the effect of **inflation**.

Payment Model

- Policy adjustments (if any) are assumed to be introduced *in order*, unless otherwise specified: 1) limit u, 2) deductible d, and finally 3) coinsurance α.

Per-Loss Y^L

Define Y^L as the amount paid per loss (can be zero), i.e.,

$$Y^L = \alpha \max\{\min(X, u) - d, 0\} = \begin{cases} 0, & X \leq d \\ \alpha(X - d), & d < X \leq u \\ \alpha(u - d), & X > u \end{cases}$$

Per-Payment Y^P

Define Y^P as the amount paid per payment, i.e.,

$$Y^P = Y^L \mid Y^L > 0$$

Policy Limit

- If only the policy limit is applied, we have

$$Y^L = Y^P = \min(X, u) = \begin{cases} X, & X \leq u \\ u, & X > u \end{cases}$$

which is a mixed distribution with a mass point at u, i.e.,

$$f_Y(y) = \begin{cases} f_X(y), & y < u \\ 1 - F_X(u), & y = u \end{cases}$$

- The expected payment amount (per-loss or per-payment) is

$$\mathbb{E}[Y] = \mathbb{E}[X \wedge u] = \int_0^u x f_X(x)\, dx + u\overline{F}_X(u) = \int_0^u \overline{F}_X(x)\, dx$$

where $\overline{F}_X(x) = 1 - F_X(x) = S_X(x)$ is the survival function.

Censored Loss

Losses above the limit u are **censored**, given that they are observed but the exact loss amounts are unknown.

Example: FAM-S Sample Q26

Example

The unlimited severity distribution for claim amounts under an auto liability insurance policy is given by the cumulative distribution:

$$F(x) = 1 - 0.8e^{-0.02x} - 0.2e^{-0.001x}, \quad x \geq 0$$

The insurance policy pays amounts up to a limit of 1000 per claim. Calculate the expected payment under this policy for one claim. [166.42]

(Exercises: FAM-S Sample Qs 6 & 14)

Increased Limit Factor

- Typically, multiple policy limits exist, but limited data is available to determine premiums for those high limits.
- Actuaries set a "lower limit" as the **basic limit**, and determine the premium for that first.
- **Increased limit factors** (ILFs) are then multiplied to determine premiums for higher limits.

Increased Limit Factor

An ILF at limit H relative to basic limit B is

$$\begin{aligned}\text{ILF}(H) &= \frac{\text{Expected Loss Cost}(H)}{\text{Expected Loss Cost}(B)} = \frac{\text{Expected Severity}(H)}{\text{Expected Severity}(B)} \\ &= \frac{\mathbb{E}[X \wedge H]}{\mathbb{E}[X \wedge B]},\end{aligned}$$

assuming the frequency is independent of the policy limit.

Example: FAM-S Sample Q26 Modified

Example

The unlimited severity distribution for claim amounts under an auto liability insurance policy is given by the cumulative distribution:

$$F(x) = 1 - 0.8e^{-0.02x} - 0.2e^{-0.001x}, \quad x \geq 0$$

The insurance policy has a basic limit of 500 per claim. The claim frequency is independent of the policy limit and there is no loading for risk or expenses.
Calculate the increased limit factor for a limit of 1000 per claim.
[1.402]

(Exercise: ASTAM F24 Q2)

Ordinary Deductible

- For an ordinary deductible d, $Y^L = \max(X - d, 0)$ with df

$$F_{Y^L}(y) = \begin{cases} F_X(d), & y = 0 \\ F_X(y + d), & y > 0 \end{cases}$$

- The expected loss cost per-loss is

$$\begin{aligned} \mathbb{E}\left[Y^L\right] &= \int_d^\infty (x - d) f_X(x)\, dx = \int_d^\infty \overline{F}_X(x)\, dx \\ &= \mathbb{E}[X] - \mathbb{E}[X \wedge d] \end{aligned}$$

- **Excess-of-loss reinsurance** with retention R pays $(X - R)_+$ per loss, so R acts like an ordinary deductible.

Truncated Loss

A **truncated loss** is not observed: with a deductible d, losses below d are truncated.

Ordinary Deductible: Per-Payment

- Loss cost per-payment $Y^P = X - d \,|\, X > d$, with

$$\overline{F}_{Y^P}(y) = \Pr(X - d > y \,|\, X > d) = \frac{\overline{F}_X(y+d)}{\overline{F}_X(d)}$$

- The expected loss cost per-payment is

$$\mathbb{E}\left[Y^P\right] = \frac{\mathbb{E}\left[Y^L\right]}{\overline{F}_X(d)} = \frac{\mathbb{E}[X] - \mathbb{E}[X \wedge d]}{\overline{F}_X(d)} = \int_d^\infty \frac{\overline{F}_X(x)}{\overline{F}_X(d)}\, dx$$

- In this case, Y^P is essentially the excess loss R.V.

Example: FAM-S Sample Q63

Example

The random variable X represents the random loss, before any deductible is applied, covered by an insurance policy. The probability density function of X is

$$f(x) = 2x, \quad 0 < x < 1.$$

Payments are made subject to a deductible, d, where $0 < d < 1$. The probability that a claim payment is less than 0.5 is equal to 0.64.
Calculate the value of d. [0.323]

Example: FAM-S Sample Q27

Example

The random variable for a loss, X, has the following characteristics:

x	$F(x)$	$\mathbb{E}[X \wedge x]$
0	0.0	0
100	0.2	91
200	0.6	153
1000	1.0	331

Calculate the mean excess loss for a deductible of 100. [300]

(Exercises: FAM-S Sample Q45 & ASTAM Sample Q15(a)(c))

Franchise Deductible

- For a **franchise deductible**, the loss payment per loss is

$$Y^L = X \cdot 1_{\{X>d\}} = \begin{cases} 0, & X \leq d \\ X, & X > d \end{cases}$$

- The expected loss cost per-loss is then

$$\mathbb{E}\left[Y^L\right] = \int_d^\infty x f_X(x)\, dx = \mathbb{E}[X] - \mathbb{E}[X \wedge d] + d \cdot \overline{F}_X(d)$$

- The expected loss cost per-payment is

$$\mathbb{E}\left[Y^P\right] = \frac{\mathbb{E}\left[Y^L\right]}{\overline{F}_X(d)} = \frac{\mathbb{E}[X] - \mathbb{E}[X \wedge d]}{\overline{F}_X(d)} + d$$

Linearly Disappearing Deductible: FAM-S Sample Q62

- **Linearly disappearing deductible**: the deductible linearly decreases from d to 0.

Example

An insurance company sells a policy with a linearly disappearing deductible such that no payment is made on a claim of 250 or less and full payment is made on a claim of 1000 or more.
Calculate the payment made by the insurance company for a loss of 700. [600]

(Exercise: FAM-S Sample Q79)

Loss Elimination Ratio

- The **loss elimination ratio** is a meaningful tool to assess the impact of policy adjustments on the loss payment.

Loss Elimination Ratio

The loss elimination ratio is the proportion of the loss (on average) that has been eliminated through the application of the policy adjustments, i.e.,

$$\mathrm{LER} = 1 - \frac{\mathbb{E}\left[Y^L\right]}{\mathbb{E}[X]}$$

For an ordinary deductible d,

$$\mathrm{LER}_d = 1 - \frac{\mathbb{E}[X] - \mathbb{E}[X \wedge d]}{\mathbb{E}[X]} = \frac{\mathbb{E}[X \wedge d]}{\mathbb{E}[X]}$$

Example: FAM-S Sample Q19

Example

You are given:

(i) Losses follow an exponential distribution with the same mean in all years.

(ii) The loss elimination ratio this year is 70%.

(iii) The ordinary deductible for the coming year is 4/3 of the current deductible.

Calculate the loss elimination ratio for the coming year. [0.799]

(Exercises: FAM-S Sample Qs 17 & 64; ASTAM S25 Q4(d)(e) & F25 Q4(a)(b))

Deductible Relativities

- Insurers often offer several deductible options and price each one relative to a **base deductible** b.
- If the number of ground-up losses does not depend on the deductible, the **deductible relativity** (deductible factor) for a deductible d is

$$D(d) = \frac{\mathbb{E}[(X-d)_+]}{\mathbb{E}[(X-b)_+]} = \frac{\mathbb{E}[X] - \mathbb{E}[X \wedge d]}{\mathbb{E}[X] - \mathbb{E}[X \wedge b]} = \frac{1 - \mathrm{LER}(d)}{1 - \mathrm{LER}(b)}$$

- The premium for deductible d is (base premium)$\times D(d)$, with $D(d) > 1$ for $d < b$ and $D(d) < 1$ for $d > b$.
- In practice, insureds who choose high deductibles may be lower risks and may stop reporting small losses, so relativities indicated by experience can differ from these model-based values.

Example: Deductible Relativities

Example

Ground-up losses follow an exponential distribution with mean 1000, and the number of losses does not depend on the deductible. The base deductible is 500.

(a) Calculate the loss elimination ratios for deductibles of 250, 500, and 1000. [0.2212, 0.3935, 0.6321]

(b) Calculate the deductible relativities for deductibles of 250 and 1000. [1.2840, 0.6065]

(Exercises: ASTAM S26 Q2(a)(b) & S25 Q3(b))

All Together

- Now we add the coinsurance factor α and consider the payment with all these adjustments.
- Further, an inflation factor r may also be added to the loss X (i.e., the predicted loss for next year is $X^* = (1+r)X$).
- The per-loss random variable Y^L is

$$Y^L = \begin{cases} 0, & X \leq \frac{d}{1+r} \\ \alpha\left[(1+r)X - d\right], & \frac{d}{1+r} \leq X < \frac{u}{1+r} \\ \alpha(u-d), & X \geq \frac{u}{1+r} \end{cases}$$

- Previous analyses are special cases, i.e., let $\alpha = 1$, $r = 0$, $d = 0$, or $u \to \infty$.

Limit & Deductible Only

- If only u and d are applied ($\alpha = 1$, $r = 0$), the per-loss random variable Y^L is

$$Y^L = \max\{\min(X, u) - d, 0\} = (X \wedge u) - (X \wedge d)$$

with distribution

$$F_{Y^L}(y) = \begin{cases} F_X(y + d), & 0 \leq y < u - d, \\ 1, & y \geq u - d. \end{cases}$$

Limit & Deductible Only (cont.)

- The expected loss cost per-loss is given by

$$\mathbb{E}\left[Y^L\right] = \mathbb{E}[X \wedge u] - \mathbb{E}[X \wedge d] = \int_d^u \overline{F}_X(x)\,dx$$

- The expected loss cost per-payment is

$$\mathbb{E}\left[Y^P\right] = \mathbb{E}\left[Y^L \,\middle|\, Y^L > 0\right] = \frac{\mathbb{E}[X \wedge u] - \mathbb{E}[X \wedge d]}{\overline{F}_X(d)}$$

$$= \int_d^u \frac{\overline{F}_X(x)}{\overline{F}_X(d)}\,dx$$

Coinsurance & Inflation

- On top of u and d, the coinsurance operates as a scaling factor:

$$F_{Y^L}(y) = \begin{cases} F_X\left(\frac{y}{\alpha} + d\right), & 0 \leq y < \alpha(u-d), \\ 1, & y \geq \alpha(u-d), \end{cases}$$

with mean

$$\mathbb{E}\left[Y^L\right] = \alpha\left(\mathbb{E}[X \wedge u] - \mathbb{E}[X \wedge d]\right) = \alpha \int_d^u \overline{F}_X(x)\, dx$$

- When all four adjustments are applied, the inflation equivalently scales down the limit and deductible, i.e.,

$$\begin{aligned} \mathbb{E}\left[Y^L\right] &= \alpha\left\{\mathbb{E}[(1+r)X \wedge u] - \mathbb{E}[(1+r)X \wedge d]\right\} \\ &= \alpha(1+r)\left\{\mathbb{E}\left[X \wedge \frac{u}{1+r}\right] - \mathbb{E}\left[X \wedge \frac{d}{1+r}\right]\right\} \end{aligned}$$

Example: FAM-S Sample Q57

Example

Loss amounts have the distribution function

$$F(x) = \begin{cases} \left(\frac{x}{100}\right)^2, & 0 \leq x \leq 100, \\ 1, & x > 100. \end{cases}$$

An insurance pays 80% of the amount of the loss in excess of an ordinary deductible of 20, subject to a maximum payment of 60 per loss.
Calculate the conditional expected claim payment, given that a payment has been made. [38.91]

(Exercise: FAM-S Sample Q59)

Example: FAM-S Sample Q33

Example

Annual prescription drug costs are modeled by a two-parameter Pareto distribution with $\theta = 2000$ and $\alpha = 2$.
A prescription drug plan pays annual drug costs for an insured member subject to the following provisions:

(i) The insured pays 100% of costs up to the ordinary annual deductible of 250.

(ii) The insured then pays 25% of the costs between 250 and 2250.

(iii) The insured pays 100% of the costs above 2250 until the insured has paid 3600 in total.

(iv) The insured then pays 5% of the remaining costs.

Calculate the expected annual plan payment. [1162.66]

(Exercise: FAM-S Sample Q60)

Example: FAM-S Sample Q36

Example

Losses in Year 1 follow a two-parameter Pareto distribution with $\alpha = 2$ and $\theta = 5$. Losses in Year 2 are uniformly 20% higher than in Year 1. An insurance covers each loss subject to an ordinary deductible of 10.
Calculate the Loss Elimination Ratio in Year 2. [0.625]

(Exercises: FAM-S Sample Qs 81 & 82)

The $(a, b, 0)$ Class

A r.v. is a member of the $(a, b, 0)$ **class** if there exist constants a and b such that

$$\frac{p_k}{p_{k-1}} = a + \frac{b}{k}, \quad k = 1, 2, 3, \ldots$$

- p_k can be determined by the law of total probability (the probabilities sum to 1).
- The mean and variance of the $(a, b, 0)$ class (see Appendix B) are

$$\mathbb{E}[N] = \frac{a+b}{1-a}, \qquad \mathrm{Var}(N) = \frac{a+b}{(1-a)^2}$$

Members of the $(a, b, 0)$ Class

- The members of the $(a, b, 0)$ class are given by

Distribution	a	b	p_0
Poisson	0	λ	$e^{-\lambda}$
Binomial	$-\frac{q}{1-q}$	$(m+1)\frac{q}{1-q}$	$(1-q)^m$
Negative binomial	$\frac{\beta}{1+\beta}$	$(r-1)\frac{\beta}{1+\beta}$	$(1+\beta)^{-r}$
Geometric $(r=1)$	$\frac{\beta}{1+\beta}$	0	$(1+\beta)^{-1}$

Members of the $(a, b, 0)$ Class (cont.)

- The recursive formula provides a graphical way to pick distributions for data fitting:

$$k\frac{p_k}{p_{k-1}} = ak + b, \quad k = 1, 2, 3, \ldots$$

  - The observed values $k\frac{\hat{p}_k}{\hat{p}_{k-1}} = k\frac{n_k}{n_{k-1}}$ should form approximately a straight line if one of the above models is appropriate.
  - The sign of a (slope) can further identify the proper model.

The $(a, b, 1)$ Class

The $(a, b, 1)$ Class

A r.v. is a member of the $(a, b, 1)$ **class** if there exist constants a and b such that

$$\frac{p_k}{p_{k-1}} = a + \frac{b}{k}, \quad k = 2, 3, \ldots$$

- At times, the $(a, b, 0)$ class does not adequately describe the characteristics of some insurance data.
- Often, the fit of any of the $(a, b, 0)$ distributions is poor for the probability of no claim/payment in a given period (i.e., p_0).
- $(a, b, 1)$ class: choose p_0 differently, but the recursion still holds for $k \geq 2$.

Zero-Truncated & Zero-Modified Distributions

- We can obtain an $(a, b, 1)$ class distribution in two ways:
  - **Zero-truncated distribution** (p_k^T): Let $p_0^T = 0$ and

$$p_k^T = \frac{1}{1 - p_0} p_k, \quad k = 1, 2, 3, \ldots$$

    where p_k is the pmf of an $(a, b, 0)$ class distribution.
  - **Zero-modified distribution** (p_k^M): choose a positive p_0^M and

$$p_k^M = \frac{1 - p_0^M}{1 - p_0} p_k = \left(1 - p_0^M\right) p_k^T, \quad k = 1, 2, 3, \ldots$$

- The kth moment of the zero-modified r.v. N^M is

$$\mathbb{E}\left[\left(N^M\right)^k\right] = \frac{1 - p_0^M}{1 - p_0} \mathbb{E}\left[N^k\right]$$

Review: Probability Generating Functions

- For a random variable N on $\{0, 1, 2, \ldots\}$, the **probability generating function** (pgf) is

$$P_N(z) = \mathbb{E}\left[z^N\right] = \sum_{k=0}^{\infty} p_k z^k.$$

- Probabilities: $p_k = P_N^{(k)}(0)/k!$; in particular, $p_0 = P_N(0)$.
- Factorial moments: $P'_N(1) = \mathbb{E}[N]$ and $P''_N(1) = \mathbb{E}[N(N-1)]$, so $\mathrm{Var}(N) = P''_N(1) + P'_N(1) - P'_N(1)^2$.
- The pgf is the mgf evaluated at $\ln z$: $P_N(z) = M_N(\ln z)$.
- If N_1 and N_2 are independent, $P_{N_1+N_2}(z) = P_{N_1}(z)\, P_{N_2}(z)$.
- If $S = X_1 + \cdots + X_N$ with i.i.d. X_i independent of N, then $P_S(z) = P_N\left(P_X(z)\right)$.

Effect of Deductibles

- When a deductible d is imposed, any losses not exceeding d will not be reported to the insurer.
- Let N^L be the number of losses and N^P be the number of payments. We have

$$N^P = I_1 + I_2 + \cdots + I_{N^L},$$

where the indicator r.v. I_j is defined as

$$I_j = \begin{cases} 1, & \text{if the } j\text{th loss results in a payment,} \\ 0, & \text{otherwise.} \end{cases}$$

- The pgf is given by

$$P_{N^P}(z) = P_{N^L}\left[P_{I_j}(z)\right] = P_{N^L}\left[1 + v(z-1)\right],$$

where $v = \Pr(X > d)$.

Frequency Adjustments

- Many frequency distributions have a pgf of the form $P_{N^L}(z) = B[\theta(z-1)]$, where the function B does not depend on θ (e.g., Poisson: $e^{\lambda(z-1)}$, so $B(x) = e^x$ and $\theta = \lambda$). Then N^P is in the same family with θ replaced by $v\theta$:

$$P_{N^P}(z) = B[\theta(1-v+vz-1)] = B[v\theta(z-1)] = P_{N^L}(z; v\theta)$$

- Parameters of the number of losses N^L and the number of payments N^P:

Distribution	N^L	N^P
Poisson	λ	$v\lambda$
Binomial	$m,\ q$	$m,\ vq$
Negative binomial	$r,\ \beta$	$r,\ v\beta$

Frequency Adjustments (cont.)

- Zero-modified versions: adjust θ as above, and revise the probability at zero by

$$\frac{1 - p_0^{M*}}{1 - p_0^*} = \frac{1 - p_0^M}{1 - p_0},$$

where $*$ marks the payment (after-deductible) distribution, and p_0, p_0^* are the unmodified probabilities of 0 with parameters θ and $v\theta$. The other probabilities and moments then follow from the zero-modified formulas on the previous slides.

Example: Exam C Spring 2007 Q39

Example

You are given:

(i) The frequency distribution for the number of losses for a policy with no deductible is negative binomial with $r = 3$ and $\beta = 5$.

(ii) Loss amounts for this policy follow the Weibull distribution with $\theta = 1000$ and $\tau = 0.3$.

Determine the expected number of payments when a deductible of 200 is applied. [8.09]

(Exercise: ASTAM Sample Q15(b))

Example: Zero-Modified

Example

You are given:

(i) The number of losses follows a zero-modified Poisson distribution with $\lambda = 0.2$ and $p_0^M = 0.4$.

(ii) Loss size follows a single-parameter Pareto distribution with $\theta = 100$ and $\alpha = 0.5$, so $\Pr(X > x) = (100/x)^{0.5}$ for $x \geq 100$.

(iii) An insurance coverage has a deductible of 250.

Calculate:

1. the probability of no paid claim. [0.6067]
2. the probability of one paid claim. [0.3689]