Stat 446
Advanced Short-Term Actuarial Modeling

Chapter 2: Aggregate Loss Models

Outline

1. Introduction
   - Aggregate Losses and Payments
2. Analytic Properties
   - Compound Distribution
   - Moment Generating Function
3. Numerical Methods
   - Recursive Methods
4. Aggregate Deductible: Stop-Loss Insurance

Aggregate Loss Models

- Two main approaches have been proposed to model the aggregate claim amount of an insurance portfolio.
  - **Collective risk model**: *macro-level* (portfolio-wise), determine the total number of claims N and the loss amount X_i of the ith claim.

$$S = \begin{cases} \sum_{i=1}^{N} X_i, & N > 0 \\ 0, & N = 0 \end{cases}$$

    Given $N = n$, the X_i's are iid r.v.'s.
  - **Individual risk model**: *micro-level*, sum of a fixed number n of insurance contracts.

$$S = X_1 + X_2 + \cdots + X_n,$$

    where X_i is the amount of loss for the ith contract, and the X_i's are independent (but not necessarily identical).

Aggregate Payment

- Under the collective risk model framework, the **aggregate payment** has two equivalent representations.
  - **Per loss** basis

$$S = \begin{cases} \sum_{i=1}^{N^L} Y_i^L, & N^L > 0, \\ 0, & N^L = 0. \end{cases}$$

  - **Per payment** basis

$$S = \begin{cases} \sum_{i=1}^{N^P} Y_i^P, & N^P > 0, \\ 0, & N^P = 0. \end{cases}$$

Distribution of S

- We first analytically examine the distributional characteristics of S.
  - The **compound distribution** of S is given by

$$\begin{aligned} F_S(x) &= \sum_{n=0}^{\infty} \Pr(N = n) \cdot \Pr(S \leq x \,|\, N = n) \\ &= p_0 + \sum_{n=1}^{\infty} p_n \Pr(X_1 + \cdots + X_n \leq x) \\ &= p_0 + \sum_{n=1}^{\infty} p_n F_X^{*n}(x), \end{aligned}$$

    where $F_X^{*n}(x) = \Pr(X_1 + \cdots + X_n \leq x)$ is the n-fold convolution of the df $F_X(x)$.

- We typically assume that N and X_i are mutually independent.

Analytic Results

- If X is continuous, the pdf of S is

$$f_S(x) = \sum_{n=1}^{\infty} p_n f_X^{*n}(x),$$

  with a discrete mass point at $x = 0$, i.e., $F_S(0) = p_0$.
- For most choices of N and X, the compound distributional values of S can only be obtained numerically.
- Examples where explicit results are available (information only):
  - Exponential severity (sum of iid exponentials is gamma)
  - Inverse Gaussian severity (sum of iid IGs is IG)
  - Poisson N (**compound Poisson distribution**)

Moments of S

- To find the moments, we first examine the MGF of S:

$$M_S(t) = \sum_{n=0}^{\infty} \mathbb{E}\left[e^{tS} \,\middle|\, N = n\right] \Pr(N = n)$$
$$= \sum_{n=0}^{\infty} p_n \left(M_X(t)\right)^n = P_N\left[M_X(t)\right]$$

- If X is discrete, the same argument gives the pgf: $P_S(z) = P_N\left[P_X(z)\right]$.
- N is the "primary" distribution and X is the "secondary".

Mean and Variance of S

$$\mathbb{E}[S] = \mathbb{E}[N]\,\mathbb{E}[X], \qquad \mathrm{Var}(S) = \mathbb{E}[N]\,\mathrm{Var}(X) + \mathrm{Var}(N)\,\mathbb{E}[X]^2$$

These follow from $\mathbb{E}[S] = \mathbb{E}\left[\mathbb{E}[S \mid N]\right]$ and $\mathrm{Var}(S) = \mathbb{E}\left[\mathrm{Var}(S \mid N)\right] + \mathrm{Var}\left(\mathbb{E}[S \mid N]\right)$.

Sum of Compound Poisson Models

- The MGF of S can be used to derive the following important theorem.

Sum of Compound Poisson Models

Suppose that S_j has a compound Poisson distribution with Poisson parameter λ_j and a severity distribution with cdf $F_j(x)$ for $j = 1, 2, \ldots, n$. Suppose also that $S_1, S_2, \ldots, S_n$ are independent. Then, $S = S_1 + \cdots + S_n$ has a compound Poisson distribution with Poisson parameter $\lambda = \lambda_1 + \cdots + \lambda_n$ and a severity distribution with cdf

$$F(x) = \sum_{j=1}^{n} \frac{\lambda_j}{\lambda} F_j(x).$$

Example: Sums of Compound Poisson Distributions

Example

Policy A has a compound Poisson distribution with $\lambda = 2$ and severity probabilities 0.6 on a payment of 1 and 0.4 on a payment of 2. Policy B has a compound Poisson distribution with $\lambda = 1$ and probabilities 0.7 on a payment of 1 and 0.3 on a payment of 3. Determine the probability that the total payment on the two policies will be 2. [0.1297]

(Exercises: ASTAM Sample Q16(b) & S25 Q2(a)-(c))

Overview

- In this section, we discuss how to approximate and compute the distribution of S.
- There are mainly two different approaches, i.e., **normal approximation** and **recursive methods**.
  - Based on the Central Limit Theorem, when $\mathbb{E}[N]$ is very large, the aggregate distribution may be well approximated using the normal distribution (Exercise: ASTAM F25 Q6).
  - Alternatively, we can construct a recursive formula to compute $\Pr(S = k)$ if the claim size distribution is discrete.
    - For continuous claim sizes, discretization of the claim size is further needed.

Normal Approximation

- In the normal approximation method, the distribution of S is assumed to be asymptotically normal.
- The approximation is reliable when the insurance portfolio is relatively large and contains independent risks.

Normal Approximation

If $\mathbb{E}[N] \to \infty$, we have

$$\frac{S - \mathbb{E}[S]}{\sqrt{\mathrm{Var}(S)}} \xrightarrow{d} N(0,1)$$

and

$$F_S(x) = \Pr\left(\frac{S - \mathbb{E}[S]}{\sqrt{\mathrm{Var}(S)}} \leq \frac{x - \mathbb{E}[S]}{\sqrt{\mathrm{Var}(S)}}\right) \approx \Phi\left(\frac{x - \mathbb{E}[S]}{\sqrt{\mathrm{Var}(S)}}\right)$$

Discrete Severity: Direct Calculation

- If the severity is discrete, the aggregate loss is also discrete. Conditioning on N,

$$g_n = \Pr(S = n) = \sum_{k=0}^{\infty} \Pr(N = k) \Pr(X_1 + \cdots + X_k = n)$$
$$= \sum_{k=0}^{\infty} p_k f_n^{*k},$$

  where $f_n = \Pr(X = n)$ and $f_n^{*k} = \Pr(X_1 + \cdots + X_k = n)$ is the k-fold **convolution** of the severity distribution (with $f_0^{*0} = 1$).
- It is possible (and convenient) to directly calculate g_n for small n (i.e., 0, 1, 2, 3), but it requires computer help for larger n.

Discrete Severity: Recursive Methods

- For the $(a, b, 0/1)$ classes, a recursive formula is available to calculate $g_k = \Pr(S = kh)$, assuming $f_j = \Pr(X = jh)$ defined on $j = 0, 1, \ldots, m$ and h is a unit length.
- For the $(a, b, 0)$ class

$$g_k = \frac{1}{1 - af_0} \sum_{j=1}^{k \wedge m} \left(a + \frac{bj}{k} \right) f_j g_{k-j}, \quad k = 1, 2, 3, \ldots$$

In particular, for a Poisson distribution N

$$g_k = \frac{\lambda}{k} \sum_{j=1}^{k \wedge m} j f_j g_{k-j}$$

Discrete Severity: Recursive Methods (cont.)

- For the general $(a, b, 1)$ class, for $k = 1, 2, 3, \ldots$

$$g_k = \frac{[p_1 - (a+b)p_0]f_k + \sum_{j=1}^{k \wedge m} \left(a + \frac{bj}{k}\right) f_j g_{k-j}}{1 - af_0}$$

- Starting point $g_0 = P_N(f_0)$ (note that $g_0 = p_0$ if $f_0 = 0$).
- **Notation**: the ASTAM formula sheet writes $f_S(x) = \Pr(S = x)$ for g_k and $f_X(y) = \Pr(X = y)$ for f_j. For example, the $(a, b, 0)$ recursion is

$$f_S(x) = \frac{1}{1 - af_X(0)} \sum_{y=1}^{x \wedge m} \left(a + \frac{by}{x}\right) f_X(y)\, f_S(x - y).$$

Example: STAM Sample Q95

Example

The number of claims in a period has a geometric distribution with mean 4. The amount of each claim X follows $\Pr(X = x) = 0.25$, $x = 1, 2, 3, 4$. The number of claims and the claim amounts are independent. S is the aggregate claim amount in the period. Calculate $F_S(3)$. [0.3456]

(Exercises: ASTAM Sample Q16(a)(c) & S25 Q2(d))

Continuous Severity: Method of Rounding

- If the severity is continuous, we discretize the distribution and then use the recursive method to compute S.
- Method of rounding:
  - Let the span $h > 0$ be the unit measurement.
  - Define f_j as the probability placed at jh for $j = 0, 1, \ldots$

$$f_0 = \Pr\left(X \leq \frac{h}{2}\right) = F_X\left(\frac{h}{2}\right)$$

and

$$\begin{aligned} f_j &= \Pr\left(jh - \frac{h}{2} < X \leq jh + \frac{h}{2}\right) \\ &= F_X\left(jh + \frac{h}{2}\right) - F_X\left(jh - \frac{h}{2}\right), \end{aligned}$$

for $j = 1, 2, \ldots$

Example: ASTAM Sample Q15(a)(b)

Example

You are given the following information on liability insurance losses.

(i) Conditioning on Λ, the ground-up loss of a liability insurance, $X|\Lambda$, follows an exponential distribution with mean $1/\Lambda$, where Λ is gamma distributed with mean 0.008 and standard deviation 0.004.

(ii) The number of losses has a negative binomial distribution with parameters $r = 2$ and $\beta = 1.5$.

(iii) An ordinary deductible of 100 is applied to each individual loss.

Example: ASTAM Sample Q15(a)(b) (cont.)

Example

(a) Show that the unconditional distribution function of the ground-up loss is

$$F_X(x) = 1 - \left(\frac{500}{500 + x}\right)^4, \quad x \geq 0.$$

(b) Determine the distribution for the number of payments. [negative binomial, $r = 2$, $\beta = 0.7234$]

Example: ASTAM Sample Q15(c)-(e)

Example

(c) Show that the distribution of the amount paid per payment, Y^P, is

$$F_{Y^P}(y) = 1 - \left(\frac{600}{600+y}\right)^4, \quad y \geq 0.$$

(d) With a span of $h = 50$, discretize the severity distribution of Y^P from part (c) using the method of rounding. You are required to calculate all values up to a discretized amount paid of 100. [$f_0 = 0.1507$, $f_1 = 0.2251$, $f_2 = 0.1552$]

(e) Use the recursive formula to calculate the discretized distribution of aggregate payments up to a discretized amount paid of 100. [$g_0 = 0.3837$, $g_1 = 0.0774$, $g_2 = 0.0651$]

(Exercises: ASTAM F23 Q1, S24 Q3, F24 Q3 & F25 Q4(c))

Continuous Severity: Local Moment Matching

- Rounding puts each interval's probability at a single point, so the discretized distribution usually does not have the same mean as X.
- **Local moment matching**: on each interval $[x_k,\, x_k + ph)$, place masses $m_0^k, \ldots, m_p^k$ at the points $x_k,\, x_k + h, \ldots, x_k + ph$ so that the probability and the first p moments of X on that interval are matched.
- With $p = 1$ and $x_k = kh$, the two masses on $[kh, (k+1)h)$ solve

$$m_0^k + m_1^k = \Pr\left(kh \le X < (k+1)h\right)$$
$$kh\, m_0^k + (k+1)h\, m_1^k = \mathbb{E}\left[X;\ kh \le X < (k+1)h\right]$$

- The probability at jh collects the masses from both neighboring intervals: $f_0 = m_0^0$ and $f_j = m_1^{j-1} + m_0^j$ for $j \ge 1$.

Continuous Severity: Local Moment Matching (cont.)

- For $p = 1$, the probabilities can be written using limited expected values:

$$f_0 = 1 - \frac{\mathbb{E}[X \wedge h]}{h},$$
$$f_j = \frac{2\,\mathbb{E}[X \wedge jh] - \mathbb{E}[X \wedge (j-1)h] - \mathbb{E}[X \wedge (j+1)h]}{h}, \quad j \geq 1$$

- The discretized distribution has the same mean as X (in general, the first p moments of X are preserved).
- Comparison with rounding:
  - Rounding is simpler and needs only F_X; local moment matching needs limited expected values but preserves moments.
  - Both converge to the true distribution as $h \to 0$; with a coarse span, moment matching usually gives better estimates of aggregate means and stop-loss premiums.

Example: Local Moment Matching

Example

Loss amounts follow an exponential distribution with mean 2. Use a span of $h = 1$.

(a) Discretize the distribution using local moment matching with one moment. Calculate f_0, f_1, and f_2. [0.2131, 0.3096, 0.1878]

(b) Repeat (a) using the method of rounding. [0.2212, 0.3064, 0.1859]

(c) Calculate the mean of each discretized distribution. [moment matching: 2; rounding: 1.979]

(Exercise: ASTAM S26 Q3(d)(e))

Aggregate Deductible: Stop-Loss Re/insurance

- An **aggregate deductible** applies to *aggregate losses* rather than individual losses.
- Stop-loss reinsurance vs. excess of loss reinsurance
  - Stop-loss is on an aggregate basis, while excess of loss is typically on a per risk/per event basis.
  - Excess of loss reinsurance may also have a coinsurance component.

Stop-Loss Insurance

A **stop-loss re/insurance** reimburses aggregate losses subject to the aggregate deductible d.
The expected insurance cost (i.e., net stop-loss premium) is

$$\mathbb{E}\left[(S-d)_+\right] = \mathbb{E}[S] - \mathbb{E}[S \wedge d]$$

Net Stop-Loss Premium

- Generally, the net stop-loss premium is

$$\mathbb{E}\left[(S-d)_+\right] = \mathbb{E}[S] - \mathbb{E}[S \wedge d] = \mathbb{E}[S] - \int_0^d \overline{F}_S(x)\,dx$$

- If S takes values only at multiples of h, with $g_k = \Pr(S = kh)$, then

$$\mathbb{E}[S \wedge d] = \sum_{k=0}^{\lfloor d/h \rfloor} kh \cdot g_k + d \cdot \Pr(S > d)$$

- When $d = mh$ is on the grid, this simplifies to

$$\mathbb{E}[S \wedge d] = \sum_{k=0}^{m-1} kh \cdot g_k + d \cdot \Pr(S \geq d)$$

(Note $\Pr(S \geq d) \neq \Pr(S > d)$ if S has a mass at d.)

Net Stop-Loss Premium (cont.)

- Alternatively, use the **area under the survival function** $\overline{F}_S(x) = \Pr(S > x)$, which is constant between grid points. For $d = mh$,

$$\mathbb{E}[S \wedge mh] = \sum_{k=0}^{m-1} h \cdot \overline{F}_S(kh),$$

  or recursively, $\mathbb{E}[S \wedge (j+1)h] = \mathbb{E}[S \wedge jh] + h \cdot \overline{F}_S(jh)$.

- For a deductible between grid points, $jh < d < (j+1)h$, the limited expected value is linear in d:

$$\mathbb{E}[S \wedge d] = \mathbb{E}[S \wedge jh] + (d - jh) \cdot \overline{F}_S(jh),$$

  so the net stop-loss premium can be found by linear interpolation between $d = jh$ and $d = (j+1)h$.

Example: FAM-S Sample Q42

Example

For a collective risk model:

(i) The number of losses has a Poisson distribution with $\lambda = 2$.

(ii) The common distribution of the individual losses is:

x	$f_X(x)$
1	0.6
2	0.4

An insurance covers aggregate losses subject to a deductible of 3. Calculate the expected aggregate payments of the insurance.
[0.7365]

(Exercises: FAM-S Sample Qs 20, 25 & 29)

Example: ASTAM Spring 2023 Q1(a)(b)

Example

You are the actuary for an insurance company that writes one year insurance coverage on a risk with the following frequency and severity distributions.

# of Losses	Probability	Amount of Losses	Probability
0	0.20	100	0.50
1	0.25	200	0.40
2	0.40	500	0.10
3	0.15		

Let S denote the aggregate losses in a year for this risk.

(a) Calculate the probability that $S \leq 350$. [0.70375]

For each policy, the premium is set at 1.15 times the expected annual claim payments.

(b) Calculate the premium for a policy with no policy modifications. [310.5]

Example: ASTAM Spring 2023 Q1(c)-(e)

Example

(c) Show that the premium for a policy with an ordinary deductible of 100 per loss is 140 to the nearest 10. You should calculate the premium to the nearest 1. [138]

(d) Calculate the premium for a policy with an aggregate deductible of 100 per insured for each year. [218.5]

(e) The insurer plans to buy stop-loss reinsurance for a policy which has no coverage modifications.

  (i) Calculate the expected annual reinsurance payment with a stop-loss deductible of 1000. [0.66]

  (ii) Explain why the insurer would prefer to use stop-loss coverage rather than setting a policy limit of 1000. [both protect the insurer, but a limit may put off policyholders]

(Exercises: ASTAM Sample Qs 7 & 16(d); S25 Q2(e) & F25 Q4(d))