Stat 446
Advanced Short-Term Actuarial Modeling

Chapter 7: Outstanding Claims Reserves

Outline

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Introduction

- In this chapter, we first develop a mathematical framework to represent the loss triangle and revisit the deterministic loss development approach learned in prior coursework.
- We then provide a probabilistic view of the run-off triangles and examine several methods as expected values: chain ladder, Bornhuetter-Ferguson, expected loss ratio, Poisson and overdispersed Poisson models, and Bayesian (credibility) methods.
- Finally, we introduce Mack's model to estimate outstanding claims reserves, which can help estimate the uncertainty/volatility of the estimation.

Run-off Triangle: Incremental Claim Payments

Accident Year	i	Development Year, j 0	1	2	3	4	5	6	7	8	9
2011	0	4360	2516	625	207	118	39	51	20	11	3
2012	1	3996	2578	449	134	49	31	31	20	4	
2013	2	3840	1738	655	175	96	40	29	14		
2014	3	5108	1757	680	216	114	69	16			
2015	4	4585	1532	414	189	80	87				
2016	5	5767	2164	410	193	86					
2017	6	5550	2540	458	252						
2018	7	6525	2828	562							
2019	8	6620	2544								
2020	9	7014									

Incremental claim payments, $X_{i,j}$

Run-off Triangle: Cumulative Claim Payments

Accident Year	i	Development Year, j									
		0	1	2	3	4	5	6	7	8	9
2011	0	4360	6876	7501	7708	7826	7865	7916	7936	7947	7950
2012	1	3996	6574	7023	7157	7206	7237	7268	7288	7292	
2013	2	3840	5578	6233	6408	6504	6544	6573	6587		
2014	3	5108	6865	7545	7761	7875	7944	7960			
2015	4	4585	6117	6531	6720	6800	6887				
2016	5	5767	7931	8341	8534	8620					
2017	6	5550	8090	8548	8800						
2018	7	6525	9353	9915							
2019	8	6620	9164								
2020	9	7014									

Cumulative claim payments, $C_{i,j}$

Notations

- Accident year cohorts: $i = 0, 1, 2, \ldots, I$.
- Development year: $j = 0, 1, 2, \ldots, J$.
- $X_{i,j}$: the incremental claim payments incurred in AY i and paid during DY j.
  - Known when $i + j \leq I$, and needs to be estimated when $i + j > I$.
- $C_{i,j}$: cumulative claims incurred in AY i and paid up to DY j.
  - $C_{i,j} = \sum_{k=0}^{j} X_{i,k}$
  - $\mathcal{D}_I = \{C_{i,j} : i + j \leq I\}$ represents the data
- **Objective**: use $\mathcal{D}_I$ to estimate $X_{i,j}$ or $C_{i,j}$ for $j \leq J$ and $i + j > I$.

Deterministic Chain Ladder

- On average, cumulative claims are assumed to follow a settlement pattern across DYs based on LDFs f_j:

$$C_{i,j+1} = C_{i,j} \cdot f_j, \quad i = 0, 1, \ldots, I, \quad j = 0, 1, \ldots, J-1$$

- Estimated development factors for each DY are

$$\hat{f}_j = \frac{\sum_{i=0}^{I-1-j} C_{i,j} \cdot f_{i,j}}{\sum_{i=0}^{I-1-j} C_{i,j}} = \frac{\sum_{i=0}^{I-1-j} C_{i,j+1}}{\sum_{i=0}^{I-1-j} C_{i,j}}, \quad \text{where } f_{i,j} = \frac{C_{i,j+1}}{C_{i,j}}$$

Deterministic Chain Ladder (cont.)

- Estimated $C_{i,k}$ and $X_{i,k}$ for $I - i + 1 \leq k \leq J$ are

$$\begin{aligned}
\hat{C}_{i,k} &= C_{i,I-i} \cdot \hat{f}_{I-i} \cdot \hat{f}_{I-i+1} \cdots \hat{f}_{k-1} \\
\hat{X}_{i,k} &= \hat{C}_{i,k} - \hat{C}_{i,k-1} \\
&= C_{i,I-i} \left(\hat{f}_{I-i} \cdot \hat{f}_{I-i+1} \cdots \hat{f}_{k-2} \right) \left(\hat{f}_{k-1} - 1 \right)
\end{aligned}$$

- (Un-)discounted reserves for AY i, $\hat{R}_i$ and $\hat{R}_i^d$, are then

$$\hat{R}_i = C_{i,I-i} \left(\hat{\lambda}_{I-i} - 1 \right), \qquad \hat{R}_i^d = \sum_{k=I-i+1}^{J} v^{(i+k-I-0.5)} \hat{X}_{i,k}$$

with $\hat{\lambda}_k = \hat{f}_k \cdot \hat{f}_{k+1} \cdots \hat{f}_{J-1}$ being the k-to-ult LDF.

Run-off Triangle: Loss Development Factors

AY	DY, j									
i	0	1	2	3	4	5	6	7	8	9
0	1.5771	1.0909	1.0276	1.0153	1.0050	1.0065	1.0025	1.0014	1.0004	–
1	1.6451	1.0683	1.0191	1.0068	1.0043	1.0043	1.0028	1.0005		
2	1.4526	1.1174	1.0281	1.0150	1.0062	1.0044	1.0021			
3	1.3440	1.0991	1.0286	1.0147	1.0088	1.0020				
4	1.3341	1.0677	1.0289	1.0119	1.0128					
5	1.3752	1.0517	1.0231	1.0101						
6	1.4577	1.0566	1.0295							
7	1.4334	1.0601								
8	1.3843									
9	–									

j	0	1	2	3	4	5	6	7	8
$\hat{f}_j$	1.43574	1.07411	1.02641	1.01226	1.00735	1.00429	1.00248	1.00099	1.00038
$\hat{\lambda}_j$	1.62722	1.13337	1.05516	1.02801	1.01556	1.00816	1.00385	1.00136	1.00038

Run-off Triangle: Aggregate Outstanding Claims

AY	DY, j										
i	0	1	2	3	4	5	6	7	8	9	$\widehat{R}_i$
0	4360	6876	7501	7708	7826	7865	7916	7936	7947	7950	0
1	3996	6574	7023	7157	7206	7237	7268	7288	7292	7295	3
2	3840	5578	6233	6408	6504	6544	6573	6587	6593	6596	9
3	5108	6865	7545	7761	7875	7944	7960	7980	7988	7991	31
4	4585	6117	6531	6720	6800	6887	6917	6934	6941	6943	56
5	5767	7931	8341	8534	8620	8683	8721	8742	8751	8754	134
6	5550	8090	8548	8800	8908	8973	9012	9034	9043	9047	247
7	6525	9353	9915	10177	10302	10377	10422	10448	10458	10462	547
8	6620	9164	9843	10103	10227	10302	10346	10372	10382	10386	1222
9	7014	10070	10817	11102	11238	11321	11370	11398	11409	11413	4399

- Aggregate outstanding claims: $\hat{R} = \sum_{i=0}^{I} \hat{R}_i = 6648$

Run-off Triangle: Discounted Outstanding Claims

AY	DY, j										Discounted
i	0	1	2	3	4	5	6	7	8	9	$\widehat{R}_i$
0	4360	2516	625	207	118	39	51	20	11	3	0
1	3996	2578	449	134	49	31	31	20	4	3	3
2	3840	1738	655	175	96	40	29	14	6	2	9
3	5108	1757	680	216	114	69	16	20	8	3	29
4	4585	1532	414	189	80	87	30	17	7	3	53
5	5767	2164	410	193	86	63	37	22	9	3	126
6	5550	2540	458	252	108	65	39	22	9	3	229
7	6525	2828	562	262	125	76	45	26	10	4	508
8	6620	2544	679	260	124	75	44	26	10	4	1143
9	7014	3056	746	286	136	83	49	28	11	4	4179

- 5% discounted aggregate outstanding claims:
  $\hat{R}^d = \sum_{i=0}^{I} \hat{R}_i^d = 6277$

Inflation-adjusted Chain Ladder

- We now consider the impact of inflation on historical and projected claims.
- Inflation can be automatically adjusted by the run-off triangle methods if the inflation rates are relatively steady.
- If inflation rates are expected to be significantly different:
  - eliminate the past inflation impact on the historical data
  - adjust the projected claim payments to reflect future inflation

Inflation-adjusted Chain Ladder (cont.)

- Let r_t be the claims inflation rate from mid-year t to mid-year $t+1$, and define the **claims inflation index** $Q_0 = 100$ and $Q_{t+1} = Q_t(1 + r_t)$.
- Inflation-adjusted incremental claims at year I values:

$$X^a_{i,j} = X_{i,j} \cdot \frac{Q_I}{Q_{i+j}}, \qquad C^a_{i,j} = \sum_{k=0}^{j} X^a_{i,k}, \quad \text{for } i + j \leq I$$

- Projected claim payments for $i + j > I$ with inflation are

$$\hat{X}_{i,j} = \hat{X}^a_{i,j} \cdot \frac{\hat{Q}_{i+j}}{Q_I}, \quad \text{where } \hat{X}^a_{i,j} = \hat{C}^a_{i,j} - \hat{C}^a_{i,j-1}$$

Example: ASTAM Fall 2023 Q5(a)(b)

Example

An insurer has a portfolio of one-year policies with annual premiums. The claims triangle below shows the incremental claim payments for the portfolio for each of the last four accident years, by development year. Assume all incremental payments are made at the middle of each year, and claims are fully developed by the end of Development Year (DY) 3.

Accident Year (AY)	**Incremental Paid Claims by Development Year (DY)**			
	0	**1**	**2**	**3**
2019	104	38	17	5
2020	106	40	19	
2021	138	45		
2022	145			

You are given the following inflation rates, measured from mid-year to mid-year:

2019-2020	2020-2021	2021-2022	2022-2023
2.0%	4.4%	7.5%	3.0%

Example: ASTAM Fall 2023 Q5(a)(b) (cont.)

Example

(a) Show that the AY 2019 inflation-adjusted aggregate paid claims, expressed in terms of mid-2023 monetary units, are 190 to the nearest 10. You should calculate the value to the nearest 0.1. [190.5]

(b) Using the inflation-adjusted chain ladder method, show that the estimated total ultimate claims from all four accident years of the table, expressed in terms of mid-2023 monetary units, is 840 to the nearest 10. You should calculate the value to the nearest 0.1. [839.4]

Example: OCR E2.2

Example

You are given a cumulative run-off triangle and historic and assumed future claim inflation rates, where the inflation rates given are assumed to run from mid-year t to mid-year $t+1$.

AY	Development Year 0	1	2	3	4
0	168,830	442,760	1,062,807	1,311,257	1,333,517
1	177,540	436,618	873,088	1,013,083	
2	203,860	499,301	1,027,061		
3	215,988	405,472			
4	191,753				

Year t	Inflation Rate (past)	Year t	Inflation Rate (assumed)
0	2.5%	4	4.0%
1	3.0%	5	4.6%
2	3.5%	6	4.0%
3	3.5%	7	3.5%

Example: OCR E2.2 (cont.)

Example

(a) Calculate the claims inflation index, Q_k, using the inflation rates, for $k = 0, 1, \ldots, 7$. Set $Q_0 = 100.0$. [100, 102.5, 105.58, 109.27, 113.10, 117.62, 123.03, 127.95; also $Q_8 = 132.43$]

(b) Calculate the inflation adjusted outstanding claims reserve. [1,926,863]

Frequency-Severity Method

- The **frequency-severity** (average cost per claim) method projects the number of claims and the cost per claim separately.
  1. Project a triangle of claim counts (e.g., claims settled) to ultimate with the chain ladder, and take the projected incremental counts $\hat{N}_{i,j}$ for $i + j > I$.
  2. Estimate the average incremental payment per settled claim in each development year from the observed cells:

$$\hat{s}_j = \frac{\sum_i X_{i,j}}{\sum_i N_{i,j}},$$

     adjusted for claims inflation if necessary.
  3. Estimate the outstanding claims as $\hat{R} = \sum_{i+j>I} \hat{N}_{i,j}\, \hat{s}_j$.
- This shows whether changes come from the number of claims or their cost, but needs reliable, consistently defined count data.
- Claims settled later tend to be larger, so each DY gets its own average cost.

Example: Frequency-Severity Method

Example

Claims are fully settled by the end of DY 2. You are given:

Cumulative settled claims

AY	0	1	2
0	50	80	90
1	60	95	
2	70		

Incremental claim payments

AY	0	1	2
0	10,000	21,000	15,000
1	12,600	24,500	
2	14,700		

(a) Project the number of claims settled in each future DY using the chain ladder. [AY 1: 11.875; AY 2: 41.364, 13.920]

(b) Calculate the average incremental payment per settled claim in each DY. [207.22, 700, 1500]

(c) Estimate the outstanding claims using the frequency-severity method. [67,648]

(d) Compare with the chain ladder applied to cumulative payments. [68,980]

(Exercise: ASTAM S24 Q1)

Probabilistic Chain Ladder

- Now we consider $\hat{C}_{i,J}$ as the expected value of the random variable $C_{i,J}$, and $\hat{f}_j$ as a function of the data $C_{i,j}$.
- We reinterpret the Chain Ladder method assumptions:
  - **CL Assumption 1**: There exist development factors f_j such that
$$\mathbb{E}\left[C_{i,j+1}|\mathcal{D}_{i+j}\right] = \mathbb{E}\left[C_{i,j+1}|C_{i,j}\right] = C_{i,j} \cdot f_j$$
  - **CL Assumption 2**: Losses are independent for different AYs, i.e., $C_{i,j}$ and $C_{l,k}$ are independent for $i \neq l$, and for all j, k.
- Testing assumptions
  - Uncorrelated development factors
  - Calendar year effects

Probabilistic Chain Ladder: Main Results

Theorem

Under the CL assumptions, we have

(i) $\hat{f}_j$ is an unbiased estimator of f_j.

(ii) $\mathbb{E}\left[\hat{f}_0 \cdot \hat{f}_1 \cdots \hat{f}_j\right] = f_0 \cdot f_1 \cdots f_j$, for any $j \leq J-1$.

More generally, the product of any sequence of $\hat{f}_j$ has a similar property. In particular

$$\mathbb{E}\left[\hat{\lambda}_j\right] = \mathbb{E}\left[\hat{f}_j \cdot \hat{f}_{j+1} \cdots \hat{f}_{J-1}\right] = f_j \cdot f_{j+1} \cdots f_{J-1} = \lambda_j$$

(iii) $\hat{C}_{i,J}|\mathcal{D}_I$ is an unbiased estimator of $C_{i,J}|\mathcal{D}_I$.

Example: ASTAM Fall 2024 Q6(a)

Example

You are analyzing a run-off triangle of aggregate claims. The development factors are:

DY, j	0	1	2	3	4	5	6	7
f_j	3.0982	1.4436	1.1955	1.0874	1.0360	1.0186	1.0056	1.0000

(a) (i) State with reasons whether this business is long tailed or short tailed. [short tailed]

(ii) You are given that the most recent cumulative claims data for Accident Year (AY) 6 is $C_{6,1} = 9{,}182$.
Show that the estimated outstanding claims for AY 6 total 9,100 to the nearest 100, using the Chain Ladder method. You should calculate the value to the nearest 1. [9104]

(Exercises: ASTAM Sample Q22 & S25 Q5(a), F25 Q5)

Probabilistic Bornhuetter-Ferguson

- The BF method: a weighted average of the Chain Ladder estimate and of a (possibly exogenous) prior estimate of the ultimate cost.
  - **BF Assumption 1**: Given β_j $(j = 0, 1, \ldots, J)$ with $\beta_J = 1$ and prior estimate of losses in AY i, μ_i $(i = 0, 1, \ldots, I)$, we have

$$\mathbb{E}\left[C_{i,0}|\mathcal{D}_{i+j}\right] = \beta_0 \cdot \mu_i$$
$$\mathbb{E}\left[C_{i,j+1}|\mathcal{D}_{i+j}\right] = C_{i,j} + \left(\beta_{j+1} - \beta_j\right)\mu_i$$

  - **BF Assumption 2**: Losses are independent for different AYs, i.e., $C_{i,j}$ and $C_{l,k}$ are independent for $i \neq l$, and for all j, k.

Probabilistic Bornhuetter-Ferguson: Main Results

- Interpretation of the parameters:
  - $\beta_j = 1/\lambda_j$: estimated proportion of ultimate claims paid by the end of DY j.
  - $\gamma_j = \beta_j - \beta_{j-1}$: estimated proportion of ultimate claims paid during DY j.

Theorem

Under the BF assumptions, the Bornhuetter-Ferguson estimate of cumulative claims is

$$\tilde{C}_{i,J} = C_{i,I-i} + \left(1 - \hat{\beta}_{I-i}\right)\mu_i = \hat{\beta}_{I-i}\hat{C}_{i,J} + \left(1 - \hat{\beta}_{I-i}\right)\mu_i$$

where $\hat{C}_{i,J} = C_{i,I-i}\hat{\lambda}_{I-i}$ is the CL estimate of cumulative claims.

- The second term, $(1 - \hat{\beta}_{I-i})\mu_i$, is the BF estimate of the **outstanding claims** for AY i.

Example: ASTAM Fall 2024 Q6(b)

Example

(b) (i) The Earned Premium in AY 6 is 18,000 and the expected loss ratio at issue is 0.85.
Show that the estimated outstanding claims for AY 6 using the Bornhuetter-Ferguson method is 7,600 to the nearest 100. You should calculate the value to the nearest 1. [7617]

(ii) Describe one advantage and one disadvantage of the Bornhuetter-Ferguson method compared with the Chain Ladder method. [less weight on immature data; but the prior is subjective]

(Exercises: ASTAM Sample Q3 & S25 Q5(b)-(e))

Expected Loss Ratio Method

- The **expected loss ratio (ELR)** method uses only an a priori estimate of the ultimate claims for AY i:

$$\mu_i = \mathrm{ELR}_i \times P_i,$$

  where P_i is the earned premium and ELR_i is the expected loss ratio (e.g., from pricing).
- The outstanding claims estimate is

$$\hat{R}_i^{\mathrm{ELR}} = \mu_i - C_{i,I-i}.$$

- It ignores how claims have developed so far, so it is mainly useful for very immature accident years or new lines with little data.
- The three methods weight the data and the prior differently:

$$\underbrace{\hat{C}_{i,J}}_{\text{CL: all data}} \quad , \qquad \underbrace{\hat{\beta}_{I-i}\,\hat{C}_{i,J} + (1-\hat{\beta}_{I-i})\,\mu_i}_{\text{BF: credibility } \hat{\beta}_{I-i}}, \qquad \underbrace{\mu_i}_{\text{ELR: all prior}}$$

Example: Expected Loss Ratio Method

Example

Continue ASTAM Fall 2024 Q6: for AY 6, $C_{6,1} = 9182$, the earned premium is 18,000, and the expected loss ratio is 0.85. The chain ladder and Bornhuetter-Ferguson estimates of the AY 6 outstanding claims are 9104 and 7617.

(a) Calculate the outstanding claims for AY 6 using the expected loss ratio method. [6118]

(b) Show that the Bornhuetter-Ferguson ultimate is a credibility-weighted average of the chain ladder and ELR ultimates, and state the credibility factor. [$\hat{\beta}_1 = 0.5021$; 16,799]

The Poisson Model

- The Poisson model: more suited to a run-off triangle of claim frequency
  - **Poisson Assumption 1**: X_{ij} are independent random variables for all i and j.
  - **Poisson Assumption 2**: There exist parameters μ_i, $i = 0, 1, \ldots, I$ and γ_j, $j = 0, 1, \ldots, J$ such that

$$X_{ij} \sim \mathrm{Poi}(\mu_i \gamma_j), \quad \mu_i, \gamma_j > 0, \quad \sum_{j=0}^{J} \gamma_j = 1$$

The Poisson Model (cont.)

- We immediately have the following results

$$\mathbb{E}[C_{i,j}] = \mu_i \sum_{k=0}^{j} \gamma_k = \mu_i \beta_j, \qquad \mathbb{E}[C_{i,J}] = \mu_i$$

$$\mathbb{E}\left[C_{i,j+1} | \mathcal{D}_{i+j}\right] = C_{i,j} + \mu_i \gamma_{j+1}$$

$$\mathbb{E}\left[C_{i,J} | \mathcal{D}_{i+j}\right] = C_{i,j} + \mu_i \sum_{k=j+1}^{J} \gamma_k$$

The Poisson Model: Relations

- The Poisson model estimate of ultimate claims for AY i

$$\hat{C}^P_{i,J} = C_{i,I-i} + \hat{\mu}_i \sum_{k=I-i+1}^{J} \hat{\gamma}_k = C_{i,I-i} + \hat{\mu}_i \left(1 - \hat{\beta}_{I-i}\right)$$

- If $\hat{\mu}_i$ is treated as a prior estimate of outstanding claims, then it follows the BF method.
- If $\hat{\mu}_i$ and γ_j are estimated from $\mathcal{D}_I$ using MLE, then it becomes the CL formula, i.e.,

$$\hat{C}^P_{i,J} = C_{i,I-i} \cdot \left(\hat{f}_{I-i} \cdot \hat{f}_{I-i+1} \cdots \hat{f}_{J-1}\right), \qquad \hat{f}_k = \frac{\sum_{j=0}^{k+1} \hat{\gamma}_j}{\sum_{j=0}^{k} \hat{\gamma}_j}$$

The Poisson Model: Relations (cont.)

- Poisson MLE model vs. Chain Ladder
  - CL: non-parametric, and offers no information about the uncertainty of the estimate
  - Poisson MLE: strong parametric assumption; the uncertainty derived can be questionable for most run-off triangle data.

Example: ASTAM Spring 2023 Q6(b)

Example

You are given the following data on cumulative claims, C_{ij}, for accident year (AY) i, $i = 0, 1, \ldots, 4$, and development year (DY) j, $j = 0, 1, \ldots, 4$.

Accident Year i	**Development Year j**				
	0	**1**	**2**	**3**	**4**
0	100	120	135	145	158
1	135	168	190	210	
2	175	210	225		
3	212	265			
4	260				

Let X_{ij} denote the incremental claims for AY i in DY j.

(b) Use the chain ladder method to calculate the projected incremental claims $\hat{X}_{23}$ and $\hat{X}_{24}$. [20.77, 22.03]

Example: ASTAM Spring 2023 Q6(c)

Example

(c) The Poisson model for outstanding claims assumes that X_{ij} are independent random variables, with

$$X_{ij} \sim \text{Poi}(\mu_i \gamma_j), \quad \mu_i, \gamma_j > 0, \quad \sum_{j=0}^{4} \gamma_j = 1.$$

(i) Show that for any given i, j, $\mathbb{E}[C_{i,j}] = \mu_i \sum_{k=0}^{j} \gamma_k$.
(ii) Interpret the μ_i parameters. [expected ultimate claims for AY i]
(iii) Interpret the γ_j parameters. [expected share of ultimate claims paid in DY j]

Example: ASTAM Spring 2023 Q6(d)(e)

Example

(d) You are given that the chain ladder method generates the same estimate of projected claims as the Poisson model, with maximum likelihood estimation.

(i) Use the values in (b) to show that $\hat{\gamma}_3 = 0.0776$ and $\hat{\gamma}_4 = 0.0823$.

(ii) Calculate the estimated standard deviation of the expected outstanding claims for AY 2. [6.542]

(iii) Using a normal approximation, calculate a 95% confidence interval for the expected outstanding claims for AY 2. [(29.98, 55.62)]

Example: ASTAM Spring 2023 Q6(d)(e) (cont.)

Example

(e) Your colleague notes that:
"The chain ladder estimate gives the same outstanding claims estimate as the Poisson model, so that the chain ladder model is the same as the Poisson model."
Critique your colleague's remarks. [same estimates, but Poisson is a stochastic model with strong assumptions; CL is deterministic]

Overdispersed Poisson Model

- In the Poisson model, $\mathrm{Var}(X_{i,j}) = \mathbb{E}[X_{i,j}]$. For claim amounts this is unrealistic, and it even depends on the monetary unit (dollars versus thousands).
- **ODP Assumption**: the $X_{i,j}$ are independent with

$$\mathbb{E}[X_{i,j}] = \mu_i \gamma_j, \qquad \mathrm{Var}(X_{i,j}) = \phi\, \mu_i \gamma_j, \qquad \sum_{j=0}^{J} \gamma_j = 1,$$

  where $\phi > 0$ is the **dispersion parameter** (equivalently, $X_{i,j}/\phi \sim \mathrm{Poi}(\mu_i \gamma_j / \phi)$).
- The estimating equations for μ_i and γ_j do not involve ϕ, so $\hat{\mu}_i$, $\hat{\gamma}_j$, and the reserves are the same as in the Poisson model, i.e., the chain ladder values. The dispersion parameter affects only the uncertainty.

Overdispersed Poisson Model (cont.)

- Estimate ϕ with the Pearson residuals of the observed cells:

$$r_{i,j} = \frac{X_{i,j} - \hat{\mu}_i \hat{\gamma}_j}{\sqrt{\hat{\mu}_i \hat{\gamma}_j}}, \qquad \hat{\phi} = \frac{1}{n-p} \sum_{i+j \le I} r_{i,j}^2,$$

  where n is the number of observed cells and p is the number of estimated parameters ($p = 2I + 1$ for a square triangle with $I + 1$ accident years).
- The outstanding claims $R = \sum_{i+j>I} X_{i,j}$ have process variance

$$\mathrm{Var}(R) = \phi \sum_{i+j>I} \mu_i \gamma_j \approx \hat{\phi}\, \hat{R}.$$

- This ignores the estimation error in $\hat{\mu}_i$ and $\hat{\gamma}_j$, so it understates the total prediction error (compare Mack's model).

Example: Overdispersed Poisson Model

Example

You are given the following incremental claim payments (in dollars). Claims are fully developed by the end of DY 3.

AY	0	1	2	3
0	100,000	60,000	25,000	8,000
1	110,000	62,000	30,000	
2	125,000	70,000		
3	130,000			

(a) Calculate $\hat{\mu}_i$ and the total outstanding claims. [193,000; 210,735; 237,134; 248,695; $\hat{R} = 169{,}564$]

(b) Estimate the dispersion parameter ϕ using Pearson residuals. [85.41]

(c) Calculate the process standard deviation of the total outstanding claims under the ODP model and under the Poisson model. [3806; 412]

Bayesian Reserving: A Credibility View

- The BF method is already a credibility (Bayesian) estimate: it gives credibility $\hat{\beta}_{I-i}$ to the chain ladder ultimate and $1-\hat{\beta}_{I-i}$ to the prior μ_i.
- In a Bayesian model, the ultimate claims of each AY are random with a prior mean μ. Each development year gives a noisy estimate of the ultimate, $X_{i,j}/\gamma_j$, with weight γ_j.
- The **Bühlmann-Straub model** gives the best linear (credibility) estimate of each AY's ultimate claims, with μ, the expected process variance v, and the variance of the hypothetical means a all estimated from the triangle.
- Unlike BF, no external (subjective) prior is needed.

Bühlmann-Straub Model for Reserves

- Start from the chain ladder: $\hat{\gamma}_j$, $\hat{\beta}_j = \sum_{k \le j} \hat{\gamma}_k$, and the ultimates $\hat{C}_{i,J} = C_{i,I-i} / \hat{\beta}_{I-i}$.
- Expected process variance, from the scatter of $X_{i,j} / \hat{\gamma}_j$ within each AY:

$$\hat{s}_i^2 = \frac{1}{I-i} \sum_{j=0}^{I-i} \hat{\gamma}_j \left(\frac{X_{i,j}}{\hat{\gamma}_j} - \hat{C}_{i,J} \right)^2, \quad i = 0, \ldots, I-1$$

  and $\hat{v} = \frac{1}{I} \sum_{i=0}^{I-1} \hat{s}_i^2$.
- Variance of the hypothetical means, from the scatter between AYs, where $\hat{\beta}_\bullet = \sum_i \hat{\beta}_{I-i}$ and $\bar{C} = \sum_i \hat{\beta}_{I-i} \hat{C}_{i,J} / \hat{\beta}_\bullet$:

$$\hat{a} = \frac{\sum_i \hat{\beta}_{I-i} \left(\hat{C}_{i,J} - \bar{C} \right)^2 - I\, \hat{v}}{\hat{\beta}_\bullet - \sum_i \hat{\beta}_{I-i}^2 / \hat{\beta}_\bullet}$$

Bühlmann-Straub Model for Reserves (cont.)

- Credibility factors, credibility-weighted mean, and estimates:

$$Z_i = \frac{\hat{\beta}_{I-i}}{\hat{\beta}_{I-i} + \hat{v}/\hat{a}}, \qquad \hat{\mu} = \frac{\sum_i Z_i \hat{C}_{i,J}}{\sum_i Z_i},$$

$$C_{i,J}^{BS} = Z_i \, \hat{C}_{i,J} + (1 - Z_i) \, \hat{\mu}$$

- Z_i increases with the development of AY i: more data means more credibility.
- If $\hat{a} \leq 0$, the data show no differences between AYs, and every AY is estimated by the overall mean.
- Compared with BF: the prior mean is estimated objectively and updates as data arrive, and the credibility reflects both v and a. However, μ is assumed to be the same for all AYs. In practice, apply the model to loss ratios or to claims per unit of exposure.

Example: Bühlmann-Straub Reserving

Example

You are given the following cumulative claims. Claims are fully developed by the end of DY 2.

AY	0	1	2
0	500	800	900
1	400	750	
2	600		

(a) Calculate the chain ladder ultimates and $\hat{\gamma}_j$. [900, 843.75, 1162.5; 0.5161, 0.3728, 0.1111]

(b) Calculate $\hat{v}$. [4363.0]

(c) Calculate $\hat{a}$. [17,170]

Example: Bühlmann-Straub Reserving (cont.)

Example

(d) Calculate the credibility factors Z_i and $\hat{\mu}$. [0.7974, 0.7777, 0.6701; 958.86]

(e) Calculate the Bühlmann-Straub estimates of the outstanding claims for AY 1 and AY 2, and compare them with the chain ladder. [119.3, 495.3; CL: 93.75, 562.5]

(Exercise: ASTAM S26 Q5(b)-(e))

Mack's Model

- Previous sections provide point estimates of outstanding claims reserves.
- It is common to add a margin for adverse experience/uncertainty in the parameter estimates.
- Mack's model extends the chain ladder method:
  - **Mack Assumption 1**: $C_{i,j}$ and $C_{l,k}$ are independent for $i \neq l$, and for all j, k.
  - **Mack Assumption 2**: For a given AY i, $\{C_{i,j}\}_j$ is a Markov chain.
  - **Mack Assumption 3**: There exist f_j such that $\mathbb{E}\left[C_{i,j+1}|C_{i,j}\right] = C_{i,j} \cdot f_j$
  - **Mack Assumption 4**: There exist σ_j^2 such that $\mathrm{Var}\left(C_{i,j+1}|C_{i,j}\right) = C_{i,j} \cdot \sigma_j^2$

Mack's Model: Results

- Mack's estimator for f_j is identical to the CL estimate:

$$\hat{f}_j = \frac{\sum_{i=0}^{I-1-j} C_{i,j} \cdot f_{i,j}}{\sum_{i=0}^{I-1-j} C_{i,j}} = \frac{\sum_{i=0}^{I-1-j} C_{i,j+1}}{\sum_{i=0}^{I-1-j} C_{i,j}}, \quad \text{where } f_{i,j} = \frac{C_{i,j+1}}{C_{i,j}}$$

- Mack's estimator for σ_j^2 is unbiased:

$$\hat{\sigma}_j^2 = \frac{1}{I-1-j} \sum_{i=0}^{I-1-j} C_{i,j} \left(f_{i,j} - \hat{f}_j \right)^2, \quad j \leq I-2$$

Mack's Model: Results (cont.)

- For $I = J$, there is not enough information for estimating $\hat{\sigma}^2_{J-1}$:

$$\hat{\sigma}^2_{J-1} = \min\left\{\hat{\sigma}^2_{J-2}, \hat{\sigma}^2_{J-3}, \hat{\sigma}^4_{J-2}/\hat{\sigma}^2_{J-3}\right\}$$

- The prediction uncertainty is then captured through the mean square error of prediction (MSEP) of $\hat{C}_{i,J}$ given $\mathcal{D}_I$.

Mack's Model: Uncertainty Measure

- MSEP of $\hat{C}_{i,J}$: **Process variance + estimation error**

$$\begin{aligned}\operatorname{MSEP}\left(\hat{C}_{i,J}\,\middle|\,\mathcal{D}_I\right) &= \mathbb{E}\left[\left(C_{i,J}-\hat{C}_{i,J}\right)^2\,\middle|\,C_{i,I-i}\right] \\ &= \operatorname{Var}\left(C_{i,J}\mid C_{i,I-i}\right)+\left(\hat{C}_{i,J}-\mathbb{E}\left[C_{i,J}\mid C_{i,I-i}\right]\right)^2\end{aligned}$$

Mack's Model: Uncertainty Measure (cont.)

- Process variance for outstanding claims is

$$\begin{aligned}\operatorname{Var}\left(C_{i,J} - C_{i,I-i} \mid C_{i,I-i}\right) &= \operatorname{Var}\left(C_{i,J} \mid C_{i,I-i}\right) \\ &\approx \hat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2 \hat{C}_{i,j}}\end{aligned}$$

- The estimation error term is

$$\left(\hat{C}_{i,J} - \mathbb{E}\left[C_{i,J} \mid C_{i,I-i}\right]\right)^2 \approx \hat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2 S_j},$$

where $S_j = \sum_{i=0}^{I-j-1} C_{i,j}$.

Mack's Model: Total Outstanding Claims

- For the total over all accident years, the MSEP is *not* just the sum of the individual MSEPs, because every AY uses the same estimated factors $\hat{f}_j$, so their estimation errors are correlated:

$$\operatorname{MSEP}\left(\sum_i \hat{C}_{i,J}\right) \approx \sum_i \operatorname{MSEP}\left(\hat{C}_{i,J}\right) + 2 \sum_{0 \le i < k \le I} \hat{C}_{i,J}\, \hat{C}_{k,J} \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2\, S_j}$$

- The inner sum runs over the development years still to come for the older AY i, which are shared with the later AY k.
- The ASTAM formula sheet gives this total MSEP as well as the MSEP for each AY.

Example: ASTAM Fall 2023 Q6

Example

You are analyzing the following cumulative claims run-off triangle information using Mack's model. Assume all claims are settled by the end of Development Year (DY) 4.

Accident Year (AY) i	**Development Year (DY)** j				
	0	1	2	3	4
0	2296	2846	3120	3284	3336
1	2183	3001	3263	3471	
2	2201	2797	3197		
3	2226	2561			
4	2456				
$\hat{f}_j$	1.2581	?	1.0583	1.0158	
$\hat{\sigma}_j$	4.3438	?	0.4465	0.1248	
S_j	8906	8644	6383	3284	

Example: ASTAM Fall 2023 Q6 (cont.)

Example

(a) Show that $\hat{f}_1 = 1.1$ to the nearest 0.1. You should calculate the value to the nearest 0.01. [1.1083]

(b) Show that the estimated total claims cost for AY 3 is $\hat{C}_{3,4} = 3050$ to the nearest 10. You should calculate the value to the nearest 0.1. [3051.2]

(c) Show that $\hat{\sigma}_1 = 1.6$ to the nearest 0.1. You should calculate the value to the nearest 0.01. [1.5975]

Example: ASTAM Fall 2023 Q6 (cont.)

Example

(d) (i) Show that the process standard deviation for outstanding claims for AY 3, using Mack's approximation, is 90 to the nearest 10. You should calculate the value to the nearest 0.1. [90.47]

(ii) Calculate the square root of the estimation error for AY 3 using Mack's approximation. [50.40]

(iii) Explain the difference between the variability described by the process variance and the variability described by the estimation error. [process: randomness of future claims; estimation: uncertainty in $\hat{f}_j$]

(Exercise: ASTAM F24 Q6(c)-(e))