1. Claims follow an exponential distribution with a mean of 500. The policy has an ordinary deductible of 100.

   (a) Calculate the loss elimination ratio for this deductible.

   Solution: $\text{LER} = \dfrac{\mathbb{E}[X \wedge 100]}{\mathbb{E}[X]} = \dfrac{500(1 - e^{-100/500})}{500} = 1 - e^{-0.2} = 0.1813.$

   (b) Calculate the expected loss amount (the expected value of the per-loss random variable).

   Solution: $\mathbb{E}[Y^L] = \mathbb{E}[X] - \mathbb{E}[X \wedge 100] = 500e^{-0.2} = 409.37.$

   (c) Now assume that the losses experience 10% inflation. What is the expected loss amount after inflation?

   Solution: Inflated losses are exponential with mean 550: $\mathbb{E}[Y^L] = 550e^{-100/550} = 458.56$.

   (d) By what percent did the expected loss amount grow? Why is that value different than 10% (give two reasons)?

   Solution: $458.56/409.37 - 1 = 12.02\%$. (1) The deductible is fixed, so the whole increase in each loss above 100 is paid by the insurer, a larger percentage of a smaller amount. (2) Some losses that were below the deductible now exceed it, so payments are made more often.

   (e) Calculate the loss elimination ratio after inflation, and explain why it changed.

   Solution: $\text{LER} = 1 - e^{-100/550} = 0.1662$. A fixed deductible eliminates a smaller share of the (larger) losses after inflation.

   (f) Ignoring inflation, the insurer also offers a deductible of 250. Using the deductible of 100 as the base, calculate the deductible relativity for the 250 deductible.

   Solution: $D(250) = \dfrac{\mathbb{E}[(X-250)_+]}{\mathbb{E}[(X-100)_+]} = \dfrac{500e^{-250/500}}{500e^{-100/500}} = e^{-0.3} = 0.7408.$

2. Claims follow a Pareto distribution with $\alpha = 4$ and $\theta = 1000$. A policy has an ordinary deductible of 100, a maximum covered loss of 1500, and a coinsurance rate of 80%. Next year, losses are subject to 10% uniform inflation, and the policy terms do not change. For next year:

   (a) What is the most the company would pay on any single claim?

   Solution: $0.8(1500 - 100) = 1120.$

   (b) What is the expected value of the per-loss random variable?

Solution: For the Pareto, $\mathbb{E}[X \wedge x] = \frac{\theta}{\alpha-1}\left[1 - \left(\frac{\theta}{\theta+x}\right)^{\alpha-1}\right]$. Deflating the deductible and limit by 1.1:

$$\mathbb{E}[Y^L] = 0.8(1.1)\left[\mathbb{E}[X \wedge 1363.64] - \mathbb{E}[X \wedge 90.91]\right] = 0.88(308.091 - 76.582) = 203.73.$$

(c) What is the expected value of the per-payment random variable?

Solution: $\Pr(1.1X > 100) = \left(\frac{1000}{1090.91}\right)^4 = 0.70607$, so $\mathbb{E}[Y^P] = 203.73/0.70607 = 288.54$.

(d) What is the expected percent of the total loss covered by the insurer?

Solution: $\mathbb{E}[1.1X] = 1.1(1000/3) = 366.67$, so the insurer covers $203.73/366.67 = 55.6\%$.

(e) Ignoring inflation, the company prices policies with no deductible and no coinsurance using a basic limit of 1500. Calculate the increased limit factor for a limit of 3000.

Solution: $\text{ILF}(3000) = \dfrac{\mathbb{E}[X \wedge 3000]}{\mathbb{E}[X \wedge 1500]} = \dfrac{328.125}{312.000} = 1.0517.$

3. Suppose that the number of claims follows a zero-modified Poisson distribution with $\lambda = 2$. Under this distribution, the mean number of claims is 1.50.

(a) Calculate the probability that the number of claims will be greater than one.

Solution: $\mathbb{E}[N] = \frac{1-p_0^M}{1-e^{-2}}\lambda$, so $\frac{1-p_0^M}{1-e^{-2}} = 0.75$ and $p_0^M = 1 - 0.75(1 - e^{-2}) = 0.3515$. Then $p_1^M = 0.75(2e^{-2}) = 0.2030$ and

$$\Pr(N > 1) = 1 - 0.3515 - 0.2030 = 0.4455.$$

(b) A deductible is introduced, and each claim now results in a payment with probability 0.4. Calculate the probability that there are no payments.

Solution: The number of payments is zero-modified Poisson with $\lambda^* = 0.4(2) = 0.8$, and $\frac{1-p_0^{M*}}{1-e^{-0.8}} = \frac{1-p_0^M}{1-e^{-2}} = 0.75$. So $p_0^{M*} = 1 - 0.75(1 - e^{-0.8}) = 0.5870$.

4. You have a portfolio of 100 independent policyholders. Each policyholder has at most one claim per year, and has a 20% chance of not having any claims for the year. Given that they do have a claim, claim amounts follow an exponential distribution with a mean of 1000.

(a) Calculate the expected aggregate loss for this portfolio.

Solution: N is binomial with $m = 100$, $q = 0.8$, so $\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X] = 80(1000) =$ 80,000.

(b) Calculate the variance of the aggregate loss.

Solution:

$$\text{Var}(S) = \mathbb{E}[N]\text{Var}(X) + \text{Var}(N)\mathbb{E}[X]^2 = 80(1000^2) + 16(1000^2) = 96{,}000{,}000.$$

(c) Assume further that you knew that 80 of the policyholders would have a claim and the other 20 would not. Calculate the expected value and the variance of the aggregate loss in this situation.

Solution: Now $N = 80$ is fixed: $\mathbb{E}[S] = 80{,}000$ and $\text{Var}(S) = 80(1000^2) = 80{,}000{,}000$. The difference, 16,000,000, is the variance due to the uncertain number of claims.

5. You sell a single auto insurance policy. The number of claims on this policy is Poisson distributed with a mean of 2. The size of each claim follows a zero-truncated Poisson distribution with $\lambda = 0.8$. S is the aggregate claim amount for this policy.

Calculate

(a) $\Pr(S = 0)$

Solution: Claim sizes are at least 1, so $\Pr(S = 0) = \Pr(N = 0) = e^{-2} = 0.1353$.

(b) $\Pr(S = 3)$

Solution: $f_X(k) = \frac{e^{-0.8}0.8^k/k!}{1-e^{-0.8}}$: $f_X(1) = 0.652773$, $f_X(2) = 0.261109$, $f_X(3) = 0.069629$. With the Poisson recursion $g_k = \frac{2}{k}\sum_{j=1}^{k} j\, f_X(j)\, g_{k-j}$:

$$\begin{aligned}
g_1 &= 2(0.652773)(0.135335) = 0.176686,\\
g_2 &= \tfrac{2}{2}\,[0.652773(0.176686) + 2(0.261109)(0.135335)] = 0.186011,\\
g_3 &= \tfrac{2}{3}\,[0.652773(0.186011) + 2(0.261109)(0.176686)\\
&\qquad +3(0.069629)(0.135335)] = 0.1613.
\end{aligned}$$

(c) the net stop-loss premium with an aggregate deductible of 2.

Solution: $\mathbb{E}[X] = \frac{0.8}{1-e^{-0.8}} = 1.452773$, so $\mathbb{E}[S] = 2.905546$. Also $\mathbb{E}[S \wedge 2] = 1(g_1) + 2(1 - g_0 - g_1) = 1.552643$, so

$$\mathbb{E}[(S-2)_+] = 2.905546 - 1.552643 = 1.3529.$$

6. You have a portfolio of 2000 calculator warranty policyholders (do not ever buy or sell these in real life). For each policyholder, the number of claims in a year is Poisson with mean 0.01, and each claim pays the calculator's replacement value. The replacement values differ as follows:

Replacement Value	Number of Policyholders
25	1000
35	800
75	200

(a) Explain why the aggregate claims S have a compound Poisson distribution, and give the Poisson parameter and the claim amount distribution.

Solution: A sum of independent compound Poisson variables is compound Poisson. $\lambda = 2000(0.01) = 20$, and the claim amount distribution is the λ-weighted mixture: 25, 35, 75 with probabilities $10/20 = 0.5$, $8/20 = 0.4$, $2/20 = 0.1$.

(b) Calculate the expected value and variance of the aggregate claims.

Solution: $\mathbb{E}[X] = 0.5(25)+0.4(35)+0.1(75) = 34$ and $\mathbb{E}[X^2] = 0.5(625)+0.4(1225)+0.1(5625) = 1365$. So $\mathbb{E}[S] = 20(34) = 680$ and $\text{Var}(S) = 20(1365) = 27{,}300$.

(c) Using the normal approximation, estimate the probability that the total claims will exceed 700.

Solution: $\Pr(S > 700) \approx 1 - \Phi\left(\frac{700-680}{\sqrt{27{,}300}}\right) = 1 - \Phi(0.121) = 0.452$.

7. The number of losses follows a Poisson distribution with mean 0.8. Loss amounts (in thousands) follow an exponential distribution with mean 3. You discretize the loss amounts with a span of $h = 1$.

(a) Using local moment matching with one moment, calculate f_0, f_1, and f_2, the probabilities of the discretized loss amounts 0, 1, and 2.

Solution: $\mathbb{E}[X \wedge x] = 3(1 - e^{-x/3})$: $\mathbb{E}[X \wedge 1] = 0.850406$, $\mathbb{E}[X \wedge 2] = 1.459749$, $\mathbb{E}[X \wedge 3] = 1.896362$.

$$f_0 = 1 - 0.850406 = 0.14959, \qquad f_1 = 2(0.850406) - 0 - 1.459749 = 0.24106,$$
$$f_2 = 2(1.459749) - 0.850406 - 1.896362 = 0.17273.$$

(b) Using the method of rounding, calculate f_0, f_1, and f_2.

Solution: With $F(x) = 1 - e^{-x/3}$: $f_0 = F(0.5) = 0.15352$, $f_1 = F(1.5) - F(0.5) = 0.23995$, $f_2 = F(2.5) - F(1.5) = 0.17193$.

(c) Using the distribution from (a) and the recursive formula, calculate the probability that the discretized aggregate losses are no more than 2.

Solution:

$$g_0 = e^{-0.8(1-0.14959)} = 0.50645, \qquad g_1 = 0.8(0.24106)(0.50645) = 0.09767,$$

$$g_2 = \tfrac{0.8}{2}\left[0.24106(0.09767) + 2(0.17273)(0.50645)\right] = 0.07940.$$

So $\Pr(S \le 2) = 0.50645 + 0.09767 + 0.07940 = 0.6835$.

(d) Which discretization preserves the mean loss? What is the mean of the discretized aggregate loss in (c)?

Solution: Local moment matching preserves the mean (3), so the discretized aggregate loss has mean $0.8(3) = 2.4$, the same as $\mathbb{E}[S]$. The rounded distribution has mean $\sum_{k\ge1} \Pr(X > k - 0.5) = \frac{e^{-1/6}}{1-e^{-1/3}} = 2.986$.

8. Assume that claim frequency (N) follows a Poisson(λ) distribution. You take a sample of 1000 policies and find 975 total claims.

(a) Find the method of moments estimate of λ.

Solution: $\hat{\lambda} = \bar{x} = 0.975$.

(b) Find the maximum likelihood estimate of λ.

Solution: $l(\lambda) = -1000\lambda + 975 \ln \lambda + \text{const}$, so $\hat{\lambda} = 975/1000 = 0.975$.

(c) Estimate the variance of the MLE, and construct an approximate 95% confidence interval for λ.

Solution: $I(\lambda) = n/\lambda$, so $\widehat{\text{Var}}(\hat{\lambda}) = 0.975/1000 = 0.000975$, and the interval is $0.975 \pm 1.96(0.03122) = (0.9138,\ 1.0362)$.

(d) Use the delta method to construct an approximate 95% confidence interval for $\Pr(N = 0)$.

Solution: $g(\lambda) = e^{-\lambda}$, so $\hat{g} = e^{-0.975} = 0.37719$ and $\widehat{\text{Var}}(\hat{g}) \approx (e^{-0.975})^2(0.000975) = 0.0001387$. The interval is $0.37719 \pm 1.96(0.01178) = (0.3541,\ 0.4003)$.

9. Ground-up losses follow an exponential distribution with mean θ. A policy has an ordinary deductible of 200 and a maximum covered loss of 2000. Losses of 200 or less are not reported. The insurer recorded six ground-up losses:

$$350 \quad 600 \quad 900 \quad 1200 \quad 2000^+ \quad 2000^+$$

where the last two losses exceeded 2000 (so the insurer paid the maximum).

(a) Write the log-likelihood function for θ.

Solution: Each loss is truncated at 200; the two censored losses contribute $\Pr(X > 2000 \mid X > 200) = e^{-1800/\theta}$. By memorylessness,

$$l(\theta) = -4\ln\theta - \frac{(150+400+700+1000)+2(1800)}{\theta} = -4\ln\theta - \frac{5850}{\theta}.$$

(b) Find the maximum likelihood estimate of θ.

Solution: $l'(\theta) = -\frac{4}{\theta} + \frac{5850}{\theta^2} = 0$ gives $\hat{\theta} = 5850/4 = 1462.5$.

(c) Using the observed information, estimate the variance of $\hat{\theta}$ and construct an approximate 95% normal confidence interval for θ.

Solution: $-l''(\hat{\theta}) = -\frac{4}{\hat{\theta}^2} + \frac{2(5850)}{\hat{\theta}^3} = \frac{4}{\hat{\theta}^2}$, so $\widehat{\mathrm{Var}}(\hat{\theta}) = \hat{\theta}^2/4 = 534{,}727$ and SD $= 731.25$. The interval is $1462.5 \pm 1.96(731.25) = (29.3,\ 2895.8)$.

(d) The normal interval in (c) can include values close to zero. Construct an approximate 95% log-transformed confidence interval for θ.

Solution: $U = \exp\left(1.96\,\mathrm{SD}(\hat{\theta})/\hat{\theta}\right) = e^{1.96(0.5)} = e^{0.98} = 2.6645$, giving $\left(\hat{\theta}/U,\ \hat{\theta}U\right) = (548.9,\ 3896.8)$.

(e) Construct an approximate 95% likelihood-based confidence interval for θ.

Solution: $l(\hat{\theta}) = -4\ln 1462.5 - 4 = -33.152$, so the interval is $\{\theta:\ l(\theta) \geq -33.152 - 1.921 = -35.072\}$. Solving $-4\ln\theta - 5850/\theta = -35.072$ numerically gives (629.6, 4710.9). Like the log-transformed interval, it is skewed to the right.

10. Losses follow an exponential distribution with mean θ. A random sample of 20 losses is distributed as follows:

Claim Size	Frequency
0–1000	13
1000–∞	7

(a) Calculate the maximum likelihood estimate of θ.

Solution: With $p = e^{-1000/\theta}$, $L = (1-p)^{13}p^7$, maximized at $\hat{p} = 7/20$. So $\hat{\theta} = -1000/\ln(0.35) = 952.5$.

(b) Calculate the maximum likelihood estimate of $\Pr(X > 2000)$.

Solution: By invariance, $e^{-2000/\hat{\theta}} = \hat{p}^2 = 0.35^2 = 0.1225$.

11. You observe the following five losses:

$$100 \quad 250 \quad 400 \quad 900 \quad 1500$$

You test H_0: the losses follow an exponential distribution with mean 600.

(a) Calculate the Kolmogorov-Smirnov test statistic.

Solution:

x	$F(x)$	$F_5(x-)$	$F_5(x)$	max difference
100	0.1535	0	0.2	0.1535
250	0.3408	0.2	0.4	0.1408
400	0.4866	0.4	0.6	0.1134
900	0.7769	0.6	0.8	0.1769
1500	0.9179	0.8	1.0	0.1179

$D = 0.1769$.

(b) The critical value at the 5% significance level is $1.36/\sqrt{n}$. State the conclusion of the test.

Solution: $1.36/\sqrt{5} = 0.608 > 0.1769$, so we do not reject H_0. (With only five observations the test has little power.)

12. Four models are fitted by maximum likelihood to 200 claim amounts:

Model	Number of Parameters	Log-likelihood
Exponential	1	−1520.4
Gamma	2	−1517.9
Weibull	2	−1518.6
Transformed Gamma	3	−1516.8

The 95th percentiles of the chi-square distribution with 1 and 2 degrees of freedom are 3.841 and 5.991.

(a) Use the likelihood ratio test at the 5% level to compare the exponential and gamma models.

Solution: $2(-1517.9+1520.4) = 5.0 > 3.841$ (1 df), so reject the exponential in favor of the gamma.

(b) Use the likelihood ratio test at the 5% level to compare the gamma and transformed gamma models.

Solution: $2(-1516.8 + 1517.9) = 2.2 < 3.841$, so the transformed gamma is not a significant improvement; keep the gamma.

(c) Which model is selected by the Akaike Information Criterion?

Solution: AIC $= l - r$: -1521.4, -1519.9, -1520.6, -1519.8. The largest is the transformed gamma.

(d) Which model is selected by the Schwarz Bayesian Criterion?

Solution: SBC $= l - \frac{r}{2} \ln 200$: -1523.05, -1523.20, -1523.90, -1524.75. The largest is the exponential; SBC penalizes extra parameters more heavily.

(e) Explain why the likelihood ratio test cannot be used to compare the gamma and Weibull models.

Solution: The likelihood ratio test requires nested models (one a special case of the other). The gamma and Weibull each have two parameters and neither is a special case of the other.