Frequency and Severity with Coverage Modifications

Chapter 8

Stat 477 - Loss Models
Introduction

In the previous weeks, we have assumed that the loss amount $X$ is also the claim amount paid.

However, there are policy modifications for which the insurer may only be liable for a portion of this loss amount.

For example:
- deductibles
- policy limits
- coinsurance

For purposes of notation, we shall denote the modified loss amount to be $Y$ and will be referred to as the claim amount paid by the insurer. More precisely, $Y^L$ denotes the per-loss variable while $Y^P$ denotes the per-payment variable.

We shall refer to $X$ as the loss amount random variable.
Policy deductible: per-loss variable

- A policy (ordinary) deductible, $d$, is a threshold amount for which it must first be exceeded by a loss before any claim can be paid.

- Once the loss $X$ exceeds this threshold, the claims paid by the insurer is $X - d$.

- The per-loss random variable is therefore

$$ Y^L = (X - d)_+ = \begin{cases} 
0, & \text{for } X \leq d \\
X - d, & \text{for } X > d 
\end{cases} $$

where $(X - d)_+$ denotes “the excess of $X$ over $d$, if positive”.

- $Y^L$ is a mixed random variable where it has a probability mass at 0 and (possibly) continuous everywhere else.

- This is exactly the same as the left censored and shifted random variable studied in Weeks 1-2.
Density, CDF, SDF of the per-loss variable

- We can write the density of $Y$ as

$$f_{Y_L}(y) = \begin{cases} 
F_X(d), & \text{for } y = 0 \\
F_X(y + d), & \text{for } y > 0 
\end{cases}.$$ 

- Its cumulative distribution function is

$$F_{Y_L}(y) = F_X(y + d),$$

and its survival function is

$$S_{Y_L}(y) = S_X(y + d).$$

- Its hazard rate function is undefined at $y = 0$. 

The random variable $Y^L$ is often referred to as the *claim amount paid per-loss event*.

Its expectation is

$$E(Y) = E[(X - d)_+] = \int_d^{\infty} (x - d) f_X(x) dx.$$ 

Using integration by parts, we can show that this is equivalent to:

$$E[(X - d)_+] = \int_d^{\infty} [1 - F_X(x)] dx.$$ 

Because there are some losses that do not produce payments at all, it is clear that $E(Y) \leq E(X)$. 


An example - Normal

Prove that if $X \sim \text{Normal}(\mu, \sigma^2)$, then the expected amount paid per loss event can be written as

$$
E[(X - d)_+] = \sigma \phi \left( \frac{d - \mu}{\sigma} \right) - (d - \mu) \left[ 1 - \Phi \left( \frac{d - \mu}{\sigma} \right) \right].
$$

Here $\phi(\cdot)$ denotes the density of a standard Normal and $\Phi(\cdot)$ denotes the cumulative distribution function of a standard Normal.
Amount paid per-payment event

- Define the random variable $Y^P$ to be the claim amount paid per payment event:
  \[ Y^P = Y^L | Y^L > 0 = Y^L | X > d. \]

- This is exactly the same as the excess loss random variable defined in Weeks 1-2.

- Its expectation, sometimes called the mean excess loss, is clearly larger than the expected amount paid per-loss event.

- Expectation:
  \[
  E(Y^P) = E(Y^L | X > d) = \int_{d}^{\infty} (x - d) f_X(x) dx \quad = \frac{E(Y^L)}{1 - F_X(d)}.
  \]
Density, CDF, SDF, hazard of the per-payment variable

- Density function: \( f_{YP}(y) = \frac{f_X(y + d)}{S_X(d)} \) for \( y > 0 \)

- Cumulative distribution function: \( F_{YP}(y) = \frac{F_X(y + d) - F_X(d)}{S_X(d)} \)

- Survival function: \( S_{YP}(y) = \frac{S_X(y + d)}{S_X(d)} \)

- Hazard rate function: \( h_{YP}(y) = \frac{f_X(y + d)}{S_X(y + d)} \).
An example - Exponential

Suppose that the loss random variable $X \sim \text{Exp}(1/\lambda)$, i.e.

$$f_X(x) = \lambda e^{-\lambda x}, \text{ for } x > 0.$$ 

Derive expressions for $E[(X - d)_+]$ and the mean excess loss $E(Y^P)$. 
Franchise deductible

- A franchise deductible modifies the ordinary deductible by adding the deductible when there is a positive amount paid.
- Once the loss $X$ exceeds the threshold $d$, the insurer pays the full loss $X$.
- The per-loss random variable for a franchise deductible is therefore

$$Y^L = \begin{cases} 0, & \text{for } X \leq d \\ X, & \text{for } X > d \end{cases}.$$

- The per-payment random variable for a franchise deductible is therefore $Y^P = X|X > d$.
- The associated density, CDF, SDF and hazards for these random variables can be found on page 182 of Klugman, et al.
Expected costs

- For an **ordinary deductible**, the expected cost per loss is
  \[E(X) - E(X \wedge d)\]
  and the expected cost per payment is
  \[\frac{E(X) - E(X \wedge d)}{1 - F_X(d)}.\]

- For a **franchise deductible**, the expected cost per loss is
  \[E(X) - E(X \wedge d) + d[1 - F_X(d)]\]
  and the expected cost per payment is
  \[\frac{E(X) - E(X \wedge d)}{1 - F_X(d)} + d.\]
Policy limits

- If the policy has a limit of say $u$, then the insurer is liable only to the extent it does not exceed $u$.
- The claim amount random variable is clearly $Y = \min(X, u)$ and can be formally written as

$$Y = (X \wedge u) = \begin{cases} 
X, & \text{for } X \leq u \\
u, & \text{for } X > u
\end{cases}.$$

- Its density can be expressed as

$$f_Y(y) = \begin{cases} 
f_X(y), & \text{for } y < u \\
1 - F_X(u), & \text{for } y = u
\end{cases}.$$

- Its distribution function can be expressed as

$$F_Y(y) = \begin{cases} 
F_X(y), & \text{for } y < u \\
1, & \text{for } y \geq u
\end{cases}.$$
Consider a policy with limit \( u \). Show that the expected cost can be expressed as

\[
E(Y) = u - \int_0^u F_X(x) \, dx = \int_0^u [1 - F_X(x)] \, dx.
\]

Calculate this expected cost in the case where the loss \( X \) has a Pareto distribution with \( \alpha = 3 \) and \( \theta = 2,000 \) for a coverage with policy limit of 3,000.
Relationships between deductibles and limits

- For any loss random variable $X$, it can be shown that
  \[ X = (X - d)_+ + (X \wedge d). \]

- The interpretation to this is intuitively clear: for a policy with a deductible $d$, losses below $d$ are not covered and therefore can be covered by another policy with a limit of $d$.

- The expectations are therefore equal:
  \[ E(X) = E[(X - d)_+] + E(X \wedge d). \]

- In the case where there is a deductible $d$, the insurer’s savings can therefore be thought of as $S = (X \wedge d)$.

- The expected savings (due to the deductible) expressed as a percentage of the loss (no deductible at all) is called the Loss Elimination Ratio:
  \[ LER = \frac{E(X \wedge d)}{E(X)}. \]
For policies with coinsurance, claim amount is proportional to the loss amount by a coinsurance factor. 

Coinsurance factor is denoted by \( \alpha \), where \( 0 < \alpha < 1 \) so that the claim payment random variable is 

\[
Y = \alpha X.
\]

Its density can be expressed as 

\[
f_Y(y) = \frac{1}{\alpha} f_X \left( \frac{y}{\alpha} \right).
\]

Its expected value is clearly \( E(Y) = \alpha E(X) \).
Combining policy modifications

- It is possible to combine deductibles and/or policy limits together with coinsurance factors.
- We shall adopt the convention that the coinsurance factor is applied after the application of any deductible or limit.
- So for example, consider the case where we have a policy deductible $d$, a policy limit $u$ and a coinsurance factor $\alpha$. The amount paid per loss random variable is given by

$$Y^L = \alpha [(X \wedge u) - (X \wedge d)].$$
An illustration

A Health Maintenance Organization (HMO) currently pays full cost of any emergency room care to its clients.
You are given that the cost of an emergency room care has an Exponential distribution with mean 1,000.
The company is evaluating the possible savings of imposing a deductible of $200 per emergency room visit, to be paid by the client.

1. Calculate the resulting loss elimination ratio due to a deductible of $200. Interpret this ratio.

2. Suppose the HMO decides to impose a per loss deductible of $200 per emergency room visit, along with a policy limit of $5,000 and a coinsurance factor of 80%. For every visit to the emergency room, calculate the expected claim amount per loss event and the expected claim amount per payment event made by the HMO.
Example 1

An insurance company offers two types of policies: Type Q and Type R. You are given:

- Type Q has no deductible, but has a policy limit of $3,000.
- Type R has no policy limit, but has a deductible of $d$.
- Losses follow a Pareto($\alpha, \theta$) distribution with $\alpha = 3$ and $\theta = 2,000$.

Calculate $d$ so that both policies have the same expected claim amount per loss.
Example 2

Well-traveled Insurance Company sells a travel insurance policy that reimburses travelers for any expenses incurred for a planned vacation that is canceled because of airline bankruptcies.

You are given:

- Individual losses follow a Pareto($\alpha, \theta$) distribution with $\alpha = 2$ and $\theta = 500$.

- Because of financial difficulties in the airline industry, Well-Traveled imposes a limit of $1,000 for each claim.

Maria’s planned vacation is canceled due to airline bankruptcies and she has incurred more than $1,000 in expenses.

Calculate the expected non-reimbursed amount of the claim.
The effect of inflation

Assume that there is a gap between the time of loss and the time the payments are made, and the insurer is obligated to cover inflation losses.

For modeling purposes, this means that instead of a loss of $X$, now the loss to consider is $(1 + r)X$, assuming an inflation rate of $r$ during the period.

Note that for policies with deductibles, the deductible is subtracted after the inflation has been taken into account. Therefore, the effect on the payment amount is greater than $100r\%$ for two reasons:

1. There are now more claims exceeding the deductible.
2. The deductible amount is usually not increased for inflation, so that those claims exceeding the deductible will increase by more than the rate of inflation, even before the inflation.
Calculating the expected claim per loss

- Assume a deductible of \( d \), a coinsurance of \( \alpha \), a policy limit of \( u \), and the inflation rate of \( r \).

- The expected claim per loss will be

  \[
  E(Y^L) = E[\alpha((1 + r)X \wedge u)] - E[\alpha((1 + r)X \wedge d)].
  \]

- This can be re-expressed as

  \[
  E(Y^L) = \alpha(1 + r) \left[ E \left( X \wedge \frac{u}{1 + r} \right) - E \left( X \wedge \frac{d}{1 + r} \right) \right].
  \]
Calculating the expected claim per payment

- We can express the probability of the loss (after inflation) exceeding the deductible, and therefore producing a claim, as follows:

\[
Pr((1 + r)X > d) = Pr\left(X > \frac{d}{1 + r}\right) = 1 - F_X\left(\frac{d}{1 + r}\right).
\]

- The expected claim amount per payment is therefore

\[
E(Y^P) = \frac{E(Y^L)}{1 - F_X\left(\frac{d}{1 + r}\right)}.
\]
Illustration

To illustrate, consider the case of the HMO in the previous example with a $200 deductible, $5,000 policy limit and an 80% coinsurance factor. Now, assume a 5% uniform inflation. Calculate the new expected claim amounts per loss and per payment.