# Aggregate Loss Models

Chapter 9

Stat 346 - Short-term Actuarial Math

# Objectives

- Individual risk model
- Collective risk model
- Computing the aggregate loss models
- Approximate methods
- Effect of policy modifications
- Chapter 9 (sections 9.1 - 9.7; exclude 9.6.1 and examples 9.9 and 9.11)
- Assignment: read section 9.7

# Individual risk model

- Consider a portfolio of $n$ insurance policies.
- Denote the loss, for a fixed period, for each policy $i$ by $X_i$, for $i = 1, \ldots, n$.
- Assume these lossses are independent and (possibly) identically distributed.
- The aggregate loss, $S$, is defined by the sum of these losses:

$$S = X_1 + X_2 + \cdots + X_n.$$

- It is possible that here a policy does not incur a loss so that each $X_i$ has a mixed distribution with a probability mass at zero.

# Alternative representation

- Assume that the losses are also identically distributed say as $X$. Then we can write $X$ as the product of a Bernoulli $I$ and a positive (continuous) random variable $Y$:

$$X = IY$$

- $I = 1$ indicates there is a claim, otherwise $I = 0$ means no claim. Let $\Pr(I = 1) = q$ and hence $\Pr(X = 0) = 1 - q$.
- In addition, assume $\mathrm{E}(Y) = \mu_Y$ and $\mathrm{Var}(Y) = \sigma_Y^2$.
- Typically it is assumed that $I$ and $Y$ are independent so that

$E(E(X|I)) = E(I E(Y))$

$$\mathrm{E}(X) = \mathrm{E}(I)\mathrm{E}(Y) = q\mu_Y$$

and $Var(X) = E(Var(X|I)) + Var(E(X|I))$, where $Var(X|I) = I\sigma_Y^2$ and $E(X|I) = I\mu_Y$

$$\mathrm{Var}(X) = q(1-q)\mu_Y^2 + q\sigma_Y^2.$$

# Mean and variance in the individual risk model

- The mean of the aggregate loss can thus be written as

$$\mathsf{E}(S) = nq\mu_Y$$

$1000\,(0.15)\left(\frac{100}{3-1}\right)$

and its variance is

$$\mathsf{Var}(S) = nq(1-q)\mu_Y^2 + nq\sigma_Y^2.$$

$1000\,(0.15)(0.85)\left(\frac{100}{2}\right)^2 + 1000\,(0.15)\,\frac{3\cdot 100^2}{1\cdot 2^2}$

- **Example:** Consider a portfolio on 1,000 insurance policies where each policy has a probability of a claim of 0.15. When a claim occurs, the amount of claim has a Pareto distribution with parameters $\alpha = 3$ and $\theta = 100$. Calculate the mean and variance of the aggregate loss.

7,500 1,443,750

# Illustrative examples

$E(S) = 0.1(1000)n = 100n$

$Var(S) = n\left((0.1)(0.9)(1000^2) + 0.1(0)\right)$

$= 90{,}000n$

$z = 1.645 = \frac{115n - 100n}{300\sqrt{n}} = \sqrt{n}\left(\frac{115-100}{300}\right)$

- **Example 1:** An insurable event has a 10% probability of occurring and when it occurs, the amount of the loss is exactly 1,000. Market research has indicated that consumers will pay at most 115 for insuring this event. How many policies must a company sell in order to have a 95% chance of making money (ignoring expenses)? Assume Normal approximation. 1083
- **Example 2:** Exercise 9.20 of textbook.

$\sqrt{n} = 32.9$

$n = 1082.41 = 1083$

# Exact distribution of $S$ using convolution

- Consider the case of $S = X_1 + X_2$. The density of $S$ can be computed using convolution as:

$$f^{*2}(s) = f_S(s) = \int_0^s f_1(s-y)f_2(y)dy = \int_0^s f_2(s-y)f_1(y)dy$$

- To extend to $n$ dimension, do it recursively. Let $f^{*(n-1)}(s)$ be the $(n-1)$-th convolution so that

$$\begin{aligned} f^{*n}(s) = f_S(s) &= \int_0^s f^{*(n-1)}(s-y)f_n(y)dy \\ &= \int_0^s f_n(s-y)f^{*(n-1)}(y)dy \end{aligned}$$

# The case of the mixed random variable

- Often, $X_i$ has a probability mass at zero so that its density function is given by
$$f_i(x) = \begin{cases} q_i, & \text{for } x = 0, \\ (1-q_i)f_Y(x), & \text{for } x > 0, \end{cases}$$
where $f_Y(\cdot)$ is a legitimate density function of a positive continuous random variable. Here, $q_i$ is interpreted as the probability that the loss is zero.
- In this case, use the cumulative distribution function:
$$F^{*2}(s) = F_S(s) = \int_0^s F_1(s-y)dF_2(y) = \int_0^s F_2(s-y)dF_1(y)$$
- To extend to $n$ dimension, do it recursively.

# The case of the Exponential distribution

Consider the case where $X_i$ has the density function expressed as

$$f_i(x) = \begin{cases} 0.10, & \text{for } x = 0, \\ 0.90 f_Y(x), & \text{for } x > 0, \end{cases}$$

where $Y$ is an Exponential with mean 1, for $i = 1, 2$.

Derive expressions for the density and distribution functions of the sum $S = X_1 + X_2$.

$F(s) = 1 - 0.99e^{-s} - 0.81se^{-s}$

$$f(s) = \begin{cases} 0.18e^{-s} + 0.81se^{-s} & s > 0 \\ 0.01 & s = 0 \end{cases}$$

# Approximating the individual risk model

- For large $n$, according to the Central Limit Theorem, $S$ can be approximated with a Normal distribution.
- Use the results on slides p. 5 to compute the mean and the variance of the aggregate loss $S$ in the individual risk model.
- Then, probabilities can be computed using Normal as follows:

$$\begin{aligned}\mathsf{Pr}(S \leq s) &= \mathsf{Pr}\left[\frac{S - \mathsf{E}(S)}{\sqrt{\mathsf{Var}(S)}} \leq \frac{S - \mathsf{E}(S)}{\sqrt{\mathsf{Var}(S)}}\right] \\ &\approx \mathsf{Pr}\left[Z \leq \frac{S - \mathsf{E}(S)}{\sqrt{\mathsf{Var}(S)}}\right] \\ &= \Phi\left(\frac{S - \mathsf{E}(S)}{\sqrt{\mathsf{Var}(S)}}\right)\end{aligned}$$

# Illustrative example 1

$E(S) = 25(0.01)(2) = 0.5$

$Var(S) = 25(0.01)(0.99)2^2 + 25(0.01)\left(\frac{20}{25}\right) = 1.19$

$E(b_k) = 2$

$E(b_k^2) = \frac{10}{25}$

An insurer has a portfolio consisting of 25 one-year life insurance policies grouped as follows:

| Insured amount $b_k$ | Number of policies $n_k$ |
|---|---|
| 1 | 10 |
| 2 | 5 |
| 3 | 10 |

$0.99^{15}\left(0.99^{10} + 10(0.01)(0.99)\right)$

$10(0.01)(0.99) + 5(0.01)(0.99)4 + 10(0.01)(0.99)9$

The probability of dying within one year is $q_k = 0.01$ for each insured, and the policies are independent.

The insurer sets up an initial capital of \$1 to cover its future obligations.

Using Normal approximation, calculate the probability that the insurer will be able to meet its financial obligation.

$\frac{1-0.5}{\sqrt{1.19}} = 0.458$

# Collective risk model

- Let $X_i$ be the claim payment made for the $i$th policyholder and let $N$ be the random number of claims. The insurer's aggregate loss is

$$S = X_1 + \cdots + X_N = \sum_{i=1}^{N} X_i.$$

- This is called the Collective Risk Model.
- If $N$, $X_1$, $X_2$, ... are independent and the individual claims $X_i$ are i.i.d., then $S$ has a *compound distribution*.
- $N$: frequency of claims; $X$: the severity of claims.
- Central question is finding the probability distribution of $S$.

# Properties of the collective risk model

- $X$ is called the individual claim and assume has moments denoted by $\mu_k = \mathsf{E}(X^k)$.
- Mean of $S$: $\mathsf{E}(S) = \mathsf{E}(X)\mathsf{E}(N) = \mu_1 \mathsf{E}(N)$
- Variance of $S$: $\mathsf{Var}(S) = \mathsf{E}(N)\mathsf{Var}(X) + \mathsf{Var}(N)\mu_1^2$
- MGF of $S$: $M_S(t) = M_N[\log M_X(t)]$
- PGF of $S$: $P_S(t) = P_N[P_X(t)]$
- CDF of $S$: $\mathsf{Pr}(S \leq s) = \sum_{n=0}^{\infty} \mathsf{Pr}(S \leq s | N = n)\mathsf{Pr}(N = n)$

# Example - Exponential/Geometric

- Suppose $X_i$ ~Exp$(1)$ and $N$ ~Geometric$(p)$.
- MGF of $N$: $M_N(t) = \dfrac{p}{1 - qe^t}$; MGF of $X$: $M_X(t) = \dfrac{1}{1-t}$
  - Mean of $S$: $\mathrm{E}(S) = q/p$
  - Variance of $S$: $\mathrm{Var}(S) = q/p + q/p^2 = (q/p)(1 + 1/p)$
  - MGF of $S$: $M_S(t) = \dfrac{p}{1 - qM_X(t)} = \dfrac{p}{1 - q/(1-t)} = p + q\dfrac{p}{p-t}$
- Observe that the mgf of $S$ can be written as a weighted average: $p\times$ mgf of r.v. $0 + q\times$ mgf of Exp$(p)$ r.v.
- Thus, $F_S(s) = p + q(1 - e^{-ps}) = 1 - qe^{-ps}$ for $s > 0$.

# Poisson number of claims

- If $N \sim \text{Poisson}(\lambda)$, so that $\lambda$ is the average number of claims, then the resulting distribution of $S$ is called a Compound Poisson.
- MGF of $N$: $M_N(t) = \exp\left[\lambda\left(e^t - 1\right)\right]$
- It can then be shown that:
  - Mean of $S$: $\mathsf{E}(S) = \lambda \mathsf{E}(X) = \lambda\mu_1$
  - Variance of $S$: $\mathsf{Var}(S) = \lambda \mathsf{E}(X^2) = \lambda\mu_2$
  - MGF of $S$: $M_S(t) = \exp\left[\lambda\left(M_X(t) - 1\right)\right]$

# Illustrative example 2

Suppose $S$ has a compound Poisson distribution with $\lambda = 0.8$ and individual claim amount distribution

| $x$ | $\Pr(X = x)$ |
|---|---|
| 1 | 0.250 |
| 2 | 0.375 |
| 3 | 0.375 |

Compute $f_S(s) = \Pr(S = s)$ for $s = 0, 1, \ldots, 6$.

# Use CLT to approximate the compound distribution

- If the average number of claims is large enough, we may use the Normal approximation to estimate the distribution of $S$.
- All you need are the mean and variance of the aggregate loss [slides p. 13]

# Compound Poisson is closed under convolution

- Theorem 9.7 of book.
- If $S_1, S_2, \ldots, S_m$ are independent compound Poisson random variables with Poisson parameters $\lambda_i$ and claim distributions (CDF) $F_i$, for $i = 1, 2, \ldots, m$, then

$$S = S_1 + S_2 + \cdots + S_m$$

  also has a compound Poisson distribution with parameter $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_m$ and individual claim (severity) distribution

$$F(x) = \sum_{i=1}^{m} \frac{\lambda_i}{\lambda} F_i(x).$$

- You add the number of claim parameters, and the individual claim distribution is a weighted sum.

# Illustrative example 3

$E(S_1) = E(E(S_1|N_1))$
$= (0.25 + 2(0.75))E(N_1) = 1.75(3) = 5.25$
$Var(S_1) = E(V(S_1|N_1)) + Var(E(S_1|N_1))$
$= E(N_1 Var(X_1)) + Var_N(N_1 \cdot E(X_1))$
$= Var(X_1)E(N) + 1.75^2 Var(N_1)$
$1.75^2(3)$

$E(S_2) = 5$
$Var(S_2) = 13.8$

You are given $S = S_1 + S_2$, where $S_1$ and $S_2$ are independent and have compound Poisson distributions with $\lambda_1 = 3$ and $\lambda_2 = 2$ and individual claim amount distributions:

| $x$ | $p_1(x)$ | $p_2(x)$ |
|---|---|---|
| 1 | 0.25 | 0.10 |
| 2 | 0.75 | 0.40 |
| 3 | 0.00 | 0.40 |
| 4 | 0.00 | 0.10 |

$Var(X_1) = 4(0.75) + 1(0.25) - 1.75^2$

Determine the mean and variance of the individual claim amount for $S$.

10.25    23.55